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Minimizing movements for mean curvature evolutions of droplets and partitions

PhD Dissertation in Mathematics

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Introduction

Dilim ilmlardan mahrum bo'lmabdi,
Bir sir qolmadiki, mavhum bo'lmabdi.
Tunu kun o'yladim yetmish ikki yil,
Angladim – hech narsa ma'lum bo'lmabdi.

Umar Hayyom

Anisotropic perimeters

$$P_\Phi(E,\Omega) = \int_{\Omega\cap\partial^* E} \Phi^o(\nu_E)\, d\mathcal{H}^n, \qquad E,\Omega \subseteq \mathbb{R}^{n+1}$$

and the area-type functionals of the form

$$\mathcal{G}(v,\widehat{\Omega}) = \int_{\widehat{\Omega}} \Phi^o(-Dv,1), \qquad \widehat{\Omega} \subseteq \mathbb{R}^{n+1}$$

where Φ^o is an anisotropy (i.e. norm), appear in many models in material science and phase transitions [65, 107], in crystal growth [5, 17, 22, 30, 31, 109], and in boundary detection and tracking [35]. Functionals like $\mathcal{G}_{\Phi^o}$, having linear growth in the gradient, appear quite frequently in calculus of variations [11, 23, 63]. The one-homogeneous case is particularly relevant, since it is related to the *anisotropic total variation functional*

$$TV_\phi(v,\widehat{\Omega}) = \int_{\widehat{\Omega}} \phi^o(Dv), \tag{1}$$

a useful functional appearing, for example, in image reconstruction and denoising [9, 37, 38, 89, 99]. Here $\phi : \mathbb{R}^n \to [0,+\infty)$ is a norm, and its dual ϕ^o is typically the restriction of Φ^o on the "horizontal" $\mathbb{R}^n$.

Minimizers of P_Φ have been widely studied [6, 107]; in particular, it is known [4, 26] that if Φ^2 is smooth and uniformly convex, (boundaries of) minimizers are smooth out of a "small" closed singular set. In contrast to the classical case, where perimeter minimizers are smooth out of a closed set of Hausdorff dimension at most $n-7$, the behaviour of minimizers of anisotropic perimeters is more irregular: for instance, there exist singular minimizing cones even for smooth and uniformly convex anisotropies in $\mathbb{R}^4$ [90]. Referring to functionals of the form (1), we recall that, if $n \le 7$, Hölder continuity of minimizers for the image denoising functional [99], consisting of the Euclidean total variation TV plus the usual quadratic fidelity term, has been studied in [34]. In [85] such result is extended to the anisotropic total variation TV_ϕ.

Anisotropic perimeters also appear in the evolution problems, such as evolutions of partitions, as a gradient flow of the (anisotropic) area functional.

Mean curvature evolution of partitions became popular in recent years because of its applications in material science and physics, especially evolutions of grain boundaries and motion of immiscible fluid systems, see e.g. [13, 28, 74, 81] and references therein. Behaviour of the Euclidean motion in the two phase case, i.e. in the case of classical motion by mean curvature of a boundary as a gradient flow of the Euclidean area functional, is rather well-understood, see for instance [16, 41, 51, 59, 61, 66, 80] and references therein. More challenging problems occur in the anisotropic case [15]: due to the possible nonregularity (for example, lack of smoothness and/or uniform convexity) of the anisotropy, one does not expect always to have a classical motion. However, short time existence of the flow for regular enough anisotropies has been obtained (via variational methods) for instance in [6] and the (crystalline) evolution of convex sets is obtained in [17]. Very recently, existence (and uniqueness) of crystalline mean curvature flow has been recently established in [39, 40]; see also [62] for the viscosity solutions of crystalline mean curvature flow. Notice that both of the existence results *highly* depend on the comparison principles.

Mean curvature evolution of interfaces in the multiphase case in general involves motion of surface junctions in $\mathbb{R}^n,$ or triple and multiple points in the plane, an already nontrivial problem. We refer to the survey [81] and references therein for recent results on curvature evolution of planar networks. Not much seems to be known in higher space dimensions; short time existence of the motion of subgraph-type partitions has been derived in [57, 58] and well-posedness and short time existence of the motion by mean curvature of three surface clusters have been recently shown in [50].

An interesting problem, related to the partitions of the space, and also mean curvature evolutions of surfaces with Neumann boundary conditions, is so called evolutions of capillary drops. Historically, capillarity problems attracted attention because of their applications in physics, for instance in the study of wetting phenomena [32, 60], energy minimizing drops and their adhesion properties [2, 29, 43, 98], as well as because of their connections with minimal surfaces, see e.g. [25, 56] and references therein. Although there are results in the literature describing the static and dynamic behaviours of droplets [3, 18, 101], not too much seems to be known concerning their mean curvature motion, especially those which flow on a horizontal hyperplane under curvature driven forces with a prescribed (possibly nonconstant) contact angle. Various results have been obtained for mean curvature flow of hypersurfaces with Dirichlet boundary conditions [67, 95, 96, 105] and zero-Neumann boundary condition [8, 64, 72, 104]. It is also worthwhile to recall that, when the contact angle is constant, the evolution can be related to the so-called mean curvature flow of surface clusters, also called space partitions (networks, in the plane).

Even in the two phase case, the classical flow describes the motion only up to the appearance of the first singularity. In order to continue the motion through singularities, several notions of generalized solutions have been suggested: Brakke varifold-solution [28], the viscosity solution (see [61] and references therein), the Almgren-Taylor-Wang [6] and Luckhaus-Sturzenhecker [78] solutions, the minimal barrier solution (see [16] and references therein); we also refer to [53, 69, 100] for other types of solutions. At the moment the lack of the comparison principle (!) in the multiphase case results in a lot of difficulties to extend such notions as viscosity (level-set) and barrier solutions, while besides Brakke solution, some other generalized solutions have been successfully extended to partitions. For example, the authors of [76] have proved the existence of a distributional solution of mean curvature evolution of partitions on the torus using the time thresholding method introduced in [86], see also [52, 87]; furthermore the authors of [73] showed the existence of a Brakke solution.

In [47] De Giorgi generalized the Almgren-Taylor-Wang and Luckhaus-Sturzenhecker approach to what he called the minimizing movements method. Let us recall the definition in [47] (see also [10, 12]).

Definition 1. *Let* S *be a topological space,* $F : S \times S \times [1,+\infty) \times \mathbb{Z} \to [-\infty,+\infty]$ *be a functional and* $u : [0,+\infty) \to S$*. We say that* u *is a generalized minimizing movement associated to* F, S *(shortly GMM) starting from* $a \in S$ *and we write* $u \in GMM(F,S,\mathbb{Z},a)$*, if there exist* $w : [1,+\infty) \times \mathbb{Z} \to S$ *and a diverging sequence* $\{\lambda_j\}$ *such that*

$$\lim_{j\to+\infty} w(\lambda_j, [\lambda_j t]) = u(t) \quad \text{for any } t \geq 0,$$

and the functions $w(\lambda,k)$*,* $\lambda \geq 1$*,* $k \in \mathbb{Z}$*, are defined inductively as* $w(\lambda,k) = a$ *for* $k \leq 0$ *and*

$$F(\lambda,k,w(\lambda,k+1),w(\lambda,k)) = \min_{s\in S} F(\lambda,k,s,w(\lambda,k)) \qquad \forall k \geq 0.$$

If $GMM(F,S,\mathbb{Z},a)$ consists of a unique element it is called a minimizing movement starting from a.

The present thesis is mainly devoted to prove the existence of minimizing movement solutions for the mean curvature flow of droplets and partitions. To our best knowledge, besides [83] there are no studies related to the GMM problem with boundary conditions. The work is divided into four chapters, the first being the preliminaries. Let us describe the main chapters that form the core of the thesis in more details.

Chapter 2: Minimizers of anisotropic perimeters with cylindrical norms

In this chapter we are interested in regularity properties of minimizers of the anisotropic perimeter

$$P_\Phi(E,\Omega) = \int_{\Omega\cap\partial^* E} \Phi^o(\nu_E)\, d\mathcal{H}^n,$$

of E in Ω, and of the related area-type functional

$$\mathcal{G}(v,\widehat{\Omega}) = \int_{\widehat{\Omega}} \Phi^o(-Dv,1).$$

Here $\Omega \subseteq \mathbb{R}^{n+1}$ is an open set, $\Phi : \mathbb{R}^{n+1} \to [0,+\infty)$ is a norm (called anisotropy), Φ^o is its dual, $E \subset \mathbb{R}^{n+1}$ is a set of locally finite perimeter, $\partial^* E$ is its reduced boundary, ν_E is the outward (generalized) unit normal to $\partial^* E$, and $\mathcal{H}^n$ is the n-dimensional Hausdorff measure in $\mathbb{R}^{n+1}$. On the other hand, $\widehat{\Omega} \subseteq \mathbb{R}^n$, v belongs to the space $BV_{\rm loc}(\widehat{\Omega})$ of functions with locally bounded total variation in $\widehat{\Omega}$, and Dv is the distributional derivative of v. When $\Omega = \widehat{\Omega} \times \mathbb{R}$ the two functionals coincide provided E is cartesian, *i.e.* E is the subgraph ${\rm sg}(v) \subset \widehat{\Omega} \times \mathbb{R}$ of the function $v \in BV_{\rm loc}(\widehat{\Omega})$ (see (2.19)).

One of the remarkable results in the classical theory of minimal surfaces is the classification of entire minimizers of the Euclidean perimeter P on $\mathbb{R}^{n+1}$: if $n \leq 6$ the only entire minimizers are hyperplanes, while for $n = 7$ there are nonlinear entire minimizers (see for instance [63, Chapter

17] and references therein); in the cartesian case (sometimes called the non parametric case), this is the well-known Bernstein problem. In the anisotropic setting, to our best knowledge, only a few results are available: entire minimizers in $\mathbb{R}^2$ are classified in [94], and minimizing cones in $\mathbb{R}^3$ for crystalline anisotropies are classified in [108]. In [71, 102] the authors show that if $n \le 2$ and Φ^2 is smooth, the only entire cartesian minimizers are the subgraphs of linear functions (anisotropic Bernstein problem), and the same result holds up to dimension $n \le 6$ if Φ is close enough to the Euclidean norm [102]. However, the anisotropic Bernstein problem seems to be still open in dimensions $3 \le n \le 6$, even for smooth and uniformly convex norms (see [97] for recent results in this direction).

The above discussion shows the difficulty of describing perimeter minimizers in the presence of an anisotropy; it seems therefore rather natural to look for reasonable assumptions on Φ that allow to simplify the classification problem. A possible requirement, which will be often (but not always) assumed in the sequel of the chapter, is that Φ is cylindrical over ϕ, *i.e.*

$$\Phi(\hat{\xi}, \xi_{n+1}) = \max\{\phi(\xi), |\xi_{n+1}|\}, \quad (\xi, \xi_{n+1}) \in \mathbb{R}^{n+1}. \tag{2}$$

Despite its splitted expression, a cylindrical anisotropy is neither smooth nor strictly convex, and this still makes the above mentioned classification rather complicated. For instance, in Examples 2.13 and 2.15 we show that there exist singular cones minimizing P_Φ in any dimension $n \ge 1$. Moreover, while it can be proved that if horizontal and vertical sections of E are minimizers of P_ϕ and P respectively then E is a minimizer of P_Φ (Remark 2.12), in general sections of a minimizer of P_Φ need not satisfy this minimality property (Examples 2.14 and 2.15).

These phenomena lead us to investigate the classification problem under some simplifying assumptions on the structure of minimizers. We shall consider two cases: cylindrical minimizers (Definition 2.16), and cartesian minimizers (Definition 2.24), the latter being our main interest. Cylindrical minimizers of P_Φ are studied in Section 2.2: in particular, in Example 2.23 we classify all cylindrical minimizers of P_Φ when $n = 2$ and the unit ball B_Φ of Φ (sometimes called Wulff shape) is a cube. Cartesian minimizers are studied in Sections 2.3, 2.4 and 2.5. In Section 2.3 we investigate the relationships between cartesian minimizers of P_Φ and minimizers of $\mathcal{G}$, provided Φ is partially monotone (Definition 2.27). In Theorem 2.32 we show that the subgraph of a minimizer of $\mathcal{G}$ is also a minimizer of P_Φ among all perturbations not preserving the cartesian structure. In particular, for Φ satisfying (2) the subgraph E of some function $u : \widehat{\Omega} \to \mathbb{R}$ is a cartesian minimizer of P_Φ in $\widehat{\Omega} \times \mathbb{R}$ if and only if u is a minimizer of TV_ϕ.

Sections 2.4 and 2.5 contain our main results, valid under the assumptions that

$$\Phi \text{ is cylindrical over } \phi \text{ and } E \text{ is cartesian.}$$

In Theorem 2.41 (see also Corollary 2.45) we prove the following Bernstein-type classification result: *if either $n \le 7$ and ϕ is Euclidean, or if $n = 2$ and ϕ^o is strictly convex, then any entire cartesian minimizer of P_Φ in $\mathbb{R}^{n+1}$ (i.e. the subgraph of a minimizer of TV_ϕ) is the subgraph of the composition of a monotone function on $\mathbb{R}$ with a linear function on $\mathbb{R}^n$.* We notice that this result is sharp: if $n = 8$, there are entire cartesian minimizers of P in $\mathbb{R}^9$ which cannot be represented as the subgraph of the composition of a monotone and a linear function (see Remark 2.44).

In view of our assumptions, also the regularity results of Section 2.5 are concerned with the anisotropic total variation functional. For our purposes, it is useful to remark that, even if the anisotropy ϕ is smooth and uniformly convex, in general minimizers of TV_ϕ are not necessarily continuous. In contrast, we remark that minimizers of TV_ϕ with continuous boundary data on bounded domains are continuous, see [70, 84, 85]. Nevertheless, in Theorems 2.47 and 2.49 we show that, *if $\phi^2 \in C^3$ is uniformly convex, then the boundary of the subgraph of a minimizer of TV_ϕ is locally Lipschitz (that is, locally a Lipschitz graph) out of a closed singular set with a suitable Hausdorff dimension depending on ϕ*. As observed in Remark 2.48, for ϕ Euclidean these statements are optimal, while the statement is false already in dimension $n = 2$ for ϕ the square norm.

Chapter 3: Minimizing movements for mean curvature flow of droplets with prescribed contact angle

If we describe the evolving droplet by a set $E(t) \subset \Omega$, $t \geq 0$ the time, where $\Omega = \mathbb{R}^n \times (0, +\infty)$ is the upper half-space in $\mathbb{R}^{n+1}$, the evolution problem we are interested in reads as

$$V = H_{E(t)} \qquad \text{on } \Omega \cap \partial E(t) \tag{3}$$

where V is the normal velocity and $H_{E(t)}$ is the mean curvature of $\partial E(t)$, supplied with the contact angle condition on the contact set (the boundary of the wetted area):

$$\nu_{E(t)} \cdot e_{n+1} = \beta \qquad \text{on } \partial E(t) \cap \partial\Omega, \tag{4}$$

where $\nu_{E(t)}$ is the outer unit normal to $\overline{\Omega \cap \partial E(t)}$ at $\partial\Omega$, and $\beta : \partial\Omega \to [-1, 1]$ is the cosine of the prescribed contact angle. We do not allow $\partial E(t)$ to be tangent to $\partial\Omega$, i.e. we suppose $|\beta| \leq 1 - 2\kappa$ on $\partial\Omega$ for some $\kappa \in (0, \frac{1}{2}]$. Following [72], in Section 3.8 we show local well-posedness of (3)-(4).

In the present chapter, the GMM scheme (see Definition 1) is applied with $S = BV(\Omega, \{0, 1\})$, $F = \mathcal{A} : BV(\Omega, \{0, 1\}) \times BV(\Omega, \{0, 1\}) \times [1, +\infty) \times \mathbb{Z} \to (-\infty, +\infty]$ defined by

$$\mathcal{A}(E, E_0, \lambda) = \mathcal{C}_\beta(E, \Omega) + \lambda \int_{E \Delta E_0} d_{E_0}\, dx,$$

where $E_0 \in BV(\Omega, \{0, 1\})$ is the initial set, d_{E_0} is the distance to $\Omega \cap \partial E_0$ and

$$\mathcal{C}_\beta(E, \Omega) = P(E, \Omega) - \int_{\partial\Omega} \beta \chi_E \, d\mathcal{H}^n$$

is the capillary functional. If $\Omega = \mathbb{R}^n$ (hence when the term $\int_{\partial\Omega} \beta\chi_E\, d\mathcal{H}^n$ is not present), the weak evolution (GMM) has been studied in [6] and [78], see also [83] for the Dirichlet case. Further when no ambiguity appears we use $GMM(E_0)$ to denote the GMM starting from $E_0 \in BV(\Omega, \{0, 1\})$.

In Section 3.1 we study the functional $\mathcal{C}(\cdot, \Omega)$ and its level-set counterpart $\mathtt{C}(\cdot, \Omega)$, including lower semicontinuity and coercivity, which will be useful in Section 3.5. In particular, the map $E \mapsto \mathcal{A}(E, E_0, \lambda)$ is $L^1(\Omega)$-lower semicontinuous if and only if $\|\beta\|_\infty \leq 1$ (Lemma 3.4). Although we can also establish the coercivity of $\mathcal{A}(\cdot, E_0, \lambda)$ (Proposition 3.1), compactness theorems in BV cannot be applied because of the unboundedness of Ω. However, in Theorem 3.12 we prove that if $E_0 \in BV(\Omega, \{0, 1\})$ is bounded and $\|\beta\|_\infty < 1$, then there is a minimizer in $BV(\Omega, \{0, 1\})$ of $\mathcal{A}(\cdot, E_0, \lambda)$, and any minimizer is bounded. In Lemma 3.17 we study the behaviour of minimizers as $\lambda \to +\infty$. In Proposition 3.15 we show existence of constrained minimizers of $\mathcal{C}_\beta(\cdot, \Omega)$, which will be used in the proof of existence of GMMs and in comparison principles. In Section 3.2 we need to generalize such existence and uniform boundedness results to minimizers of functionals of type $\mathcal{C}_\beta(\cdot,\Omega)+\mathcal{V}$ under suitable hypotheses on $\mathcal{V}$.

In Section 3.4 we study the regularity of minimizers $\mathcal{A}(\cdot,E_0,\lambda)$ (Theorem 3.22). We point out the uniform density estimates for minimizers of $\mathcal{A}(\cdot,E_0,\lambda)$ and constrained minimizers of $\mathcal{C}_\beta(\cdot,\Omega)$ (Theorem 3.20 and Proposition 3.26), which are the main ingredients in the existence proof of GMMs (Section 3.6), and in the proof of coincidence with distributional solutions (Section 3.7).

In Section 3.5 we prove the following comparison principle for minimizers of $\mathcal{A}(\cdot,E_0,\lambda)$ (Theorem 3.27): *if* E_0, F_0 *are bounded,* $E_0\subseteq F_0$, $\|\beta_1\|_\infty, \|\beta_2\|_\infty<1$ *and* $\beta_1\le\beta_2$, *then*

a) there exists a minimizer F_λ^* *of* $\mathcal{A}_{\beta_2}(\cdot,F_0,\lambda)$ *containing any minimizer of* $\mathcal{A}_{\beta_1}(\cdot,E_0,\lambda)$;

b) there exists a minimizer $E_{\lambda *}$ *of* $\mathcal{A}_{\beta_1}(\cdot,E_0,\lambda)$ *contained in any minimizer of* $\mathcal{A}_{\beta_2}(\cdot,F_0,\lambda)$;

if in addition $\operatorname{dist}(\Omega\cap\partial E_0,\Omega\cap\partial F_0)>0$, *then any minimizer* E_λ *and* F_λ *of* $\mathcal{A}_{\beta_1}(\cdot,E_0,\lambda)$ *and* $\mathcal{A}_{\beta_2}(\cdot,F_0,\lambda)$ *respectively, satisfy* $E_\lambda\subseteq F_\lambda$. As a corollary, we show that if E^+ is a bounded minimizer of $\mathcal{C}_\beta(\cdot,\Omega)$ in the collection $\mathcal{E}(E^+)$ of all finite perimeter sets containing E^+, and if $\|\beta\|_\infty<1$, then for any $E_0\subseteq E^+$, a minimizer E_λ of $\mathcal{A}_\beta(\cdot,E_0,\lambda)$ satisfies $E_\lambda\subseteq E^+$ (Proposition 3.37).

In Section 3.6 we apply the scheme in Definition 1 to the functional $\mathcal{A}_\beta(\cdot,E_0,\lambda)$: as in [78, 91] we build a locally $\frac12$-Hölder continuous generalized minimizing movement $t\in[0,+\infty)\mapsto E(t)\in BV(\Omega,\{0,1\})$ starting from a bounded set $E_0\in BV(\Omega,\{0,1\})$ (Theorem 3.38). Moreover, using the results of Section 3.5, we prove that any GMM starting from a bounded set stays bounded. In general, for two GMMs one cannot expect a comparison principle (for example in the presence of fattening). However, the notions of *maximal* and *minimal* GMMs (Definition 3.39) are always comparable if the initial sets are comparable (Theorem 3.40). This requires regularity of minimizers of $\mathcal{A}_\beta(\cdot,E_0,\lambda)$ and $\mathcal{C}_\beta(\cdot,\Omega)$, see Sections 3.3 and 3.4. Finally, in Section 3.7 we prove that, under a suitable conditional convergence assumption and if $1\le n\le 6$, our GMM solution is, in fact, a *distributional solution* to (3)-(4).

Chapter 4: Minimizing movements for partitions

Applying Definition 1 for $S=\mathbb{P}_b(N+1)$, the collection of all partitions of $\mathbb{R}^n$, $n\ge 2$, having $N+1\ge 2$ components, with the first N-components bounded, endowed with the $L^1(\mathbb{R}^n)$-topology, and following [48],

$$F_H^\Phi(\mathcal{A},\mathcal{B};\lambda)=\operatorname{Per}_\Phi(\mathcal{A})+\sum_{j=1}^{N+1}\int_{A_j}H_j dx+\frac{\lambda}{2}\sum_{j=1}^{N+1}\int_{A_j\Delta B_j}d(x,\partial B_j)dx,\quad \mathcal{A},\mathcal{B}\in\mathbb{P}_b(N+1)$$

where $\operatorname{Per}_\Phi(\mathcal{A})=\sum\limits_{j=1}^{N+1}P_{\phi_j}(A_j)$ is the anisotropic petimeter of the partition $\mathcal{A}=(A_1,\dots,A_{N+1})$, $d(\cdot,E)$ is the distance function from $E\subseteq\mathbb{R}^n$, $\mathcal{B}\in\mathbb{P}_b(N+1)$ is an initial partition, and H_i, $i=1,\dots,N+1$, are suitable driving forces, in the present chapter, we prove the existence of a generalized minimizing movement solution in $\mathbb{P}_b(N+1)$. This is the multiphase generalization of the evolution of a compact boundary in the two-phase case ($N=1$), for which the generalized minimizing movement solution has been introduced and studied in [6, 78].

Our main result is the following (see Theorem 4.16 and Corollary 4.17 for the precise statements):

Theorem 1. *Suppose that* $H_j \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$, $j = 1, \dots, N$, *for some* $p > n$ *and* $H_{N+1} \in L^1(\mathbb{R}^n)$ *are such that*

$$H_j \geq H_{N+1} \quad \text{a.e. in } \mathbb{R}^n \setminus B_R(0), \qquad j = 1, \dots, N$$

and $\Phi = (\phi_1, \dots, \phi_{N+1})$ *satisfies*

$$\sup_j \sup_\xi \phi^o(\xi) < \frac{N+2}{N} \inf_j \inf_\xi \phi^o(\xi).$$

Then for any $\mathcal{G} \in \mathbb{P}_b(N+1)$, $GMM(F_H^\Phi, \mathcal{G})$ *is nonempty, i.e. there exists a generalized minimizing movement starting from* $\mathcal{G}$. *Moreover,*

1) *any such movement* $\mathcal{M}(t) = (M_1(t), \dots, M_{N+1}(t))$ *is locally* $\frac{1}{n+1}$*-Hölder continuous in time;*

2) $\bigcup_{j=1}^N M_j(t)$ *is contained in the closed convex envelope of the union* $\bigcup_{j=1}^N G_j \cup B_R(0)$ *for any* $t > 0$.

In particular, if $H_j \equiv 0$ $\forall j$, *then* $\bigcup_{j=1}^N M_j(t)$ *is contained in the closed convex envelope of bounded components of* $\mathcal{G}$.

To prove Theorem 1 we establish uniform density estimates for minimizers of F_H^Φ. A lower-type density estimate for minimizers of F could be proven using the slicing method for currents as in the thesis [33], or also using the infiltration technique of [77, Lemma 4.6] (see also [79, Section 30.2]). In Section 4.1 we prove that (Λ, r_0)-minimizers of Per in $\mathbb{R}^n$ (Definition 4.5) satisfy uniform density estimates using the method of cutting out and filling in with balls, an argument of [78].

In Theorems 4.22 and 4.24 we also show the following consistency and stability result.

Theorem 2. *Suppose that all entries of* Φ *are Euclidean and* $\mathcal{C} \in \mathbb{P}_b(N+1)$ *is such that* $C_1, \dots, C_N$ *are convex sets with disjoint closures. Then the generalized minimizing movement associated to* F *and starting from* $\mathcal{C}$ *is a minimizing movement* $\{\mathcal{M}\} = MM(F, \mathcal{C})$ *and writing*

$$\mathcal{M}(t) = (M_1(t), \dots, M_{N+1}(t)),$$

we have that each $M_i(t)$ *agrees with the classical mean curvature flow starting from* C_i, *up to the extinction time. Moreover, if a sequence* $\{\mathcal{G}^{(k)}\} \subset \mathbb{P}_b(N+1)$ *converges to* $\mathcal{C} \in \mathbb{P}_b(N+1)$ *in the Hausdorff distance, then any* $\mathcal{M}^{(k)} \in GMM(F, \mathcal{G}^{(k)})$ *converges to* $\{\mathcal{M}\} = MM(F, \mathcal{C})$ *in the Hausdorff distance.*

The proof of the consistency with the classical mean curvature flow relies on the results of [17], while for the stability in the Hausdorff distance we employ the comparison results of Chapter 3.

Chapter 1

Notation and preliminaries

In this chapter we introduce the notation and collect some important properties of sets of finite perimeter. The standard references for BV-functions and sets of finite perimeter are [11, 63, 79].

We use $\mathbb{N}_0$ to denote the set of all nonnegative integers. Given a finite subset $I \subset \mathbb{N}_0$, we write $|I|$ for the number of elements of I. Given an open set $\Omega \subseteq \mathbb{R}^m$, $\mathrm{Op}(\Omega)$ (resp. $\mathrm{Op}_b(\Omega)$) stands for the collection of open (resp. bounded open) subsets of Ω; χ_F stands for the characteristic function of the Lebesgue measurable set $F \subseteq \mathbb{R}^m$ and $|F|$ for its Lebesgue measure; $F^c := \mathbb{R}^m \setminus F$ and $\omega_m := |B_1(0)|$, where $B_r(x) \subset \mathbb{R}^m$ is the ball of radius $r > 0$ centered at x. We write A° to denote the interior of A. The sequence $\{E_k\}$ converges to E in $L^1(\Omega)$ (resp. $L^1_{\mathrm{loc}}(\Omega)$) if $|E_k \Delta E| \to 0$ (resp. $|(E_k \Delta E) \cap V| \to 0$ for any $V \in \mathrm{Op}_b(\Omega)$) as $k \to +\infty$. Recall that $u \in L^1(\Omega)$ is said to have bounded variation if Du is a Radon measure of bounded variation, i.e.

$$\int_\Omega |Du| := \sup\left\{ -\int_\Omega u(x)\,\mathrm{div}\,\phi(x)\,dx : \ \phi \in C^1_c(\Omega;\mathbb{R}^m),\ |\phi| \le 1 \right\} < +\infty. \tag{1.1}$$

The set of $L^1(\Omega)$ resp. $L^1_{\mathrm{loc}}(\Omega)$ functions having bounded variation resp. locally bounded variation (i.e. $\int_U |Du| < +\infty$ whenever $U \in \mathrm{Op}_b(\Omega)$) in Ω is denoted by $BV(\Omega)$ resp. $BV_{\mathrm{loc}}(\Omega)$ and

$$BV(\Omega,\{0,1\}) := \{E \subseteq \Omega : \ \chi_E \in BV(\Omega)\},$$

resp.

$$BV_{\mathrm{loc}}(\Omega,\{0,1\}) := \{E \subseteq \Omega : \ \chi_E \in BV_{\mathrm{loc}}(\Omega)\}.$$

1.1 Sets of finite perimeter

Given $E \subseteq BV(\Omega,\{0,1\})$ we denote by $P(E,\Omega)$ the *perimeter* of E in Ω, i.e. $P(E,\Omega) := \int_\Omega |D\chi_E|$. We say that $E \subset \Omega$ has locally finite perimeter in Ω if $E \in BV_{\mathrm{loc}}(\Omega,\{0,1\})$. For simplicity, we set $P(E) := P(E,\mathbb{R}^m)$ provided $E \in BV(\mathbb{R}^m,\{0,1\})$. Further, given a Lebesgue measurable set $E \subseteq \mathbb{R}^m$ and $\alpha \in [0,1]$ we define

$$E^{(\alpha)} := \left\{ x \in \mathbb{R}^m : \ \lim_{\rho \to 0^+} \frac{|B_\rho(x) \cap E|}{|B_\rho(x)|} = \alpha \right\}.$$

Unless otherwise stated, we always suppose that any locally finite perimeter set E we consider coincides with $E^{(1)}$ (so that by [63, Proposition 3.1] ∂E coincides with the topological boundary, see also [55]). We recall that $\overline{\partial^* E} = \partial E$ and $D\chi_E = \nu_E d\mathcal{H}^{m-1} \llcorner \partial^* E$, where $\mathcal{H}^k$ is the k-dimensional Hausdorff measure in $\mathbb{R}^m$ and $\llcorner$ is the symbol of restriction.

Given a nonempty set $E \subseteq \mathbb{R}^n$, $d(\cdot, E)$ stands for the distance function from E and

$$\tilde{d}(x, \partial E) = d(x, E) - d(x, \mathbb{R}^n \setminus E)$$

is the signed distance function from ∂E, negative inside E.

By [44, Theorem II], for every $E \in BV_{\text{loc}}(\Omega, \{0,1\})$ the additive set function $O \mapsto \int_O |D\chi_E|$ defined on the open sets $O \subseteq \Omega$ extends to a measure $B \mapsto \int_B |D\chi_E|$ defined on the Borel σ-algebra of Ω. Moreover, $P(\cdot, \Omega)$ is submodular, i.e.

$$P(E \cap F, \Omega) + P(E \cup F, \Omega) \le P(E, \Omega) + P(F, \Omega) \quad \text{for any } E, F \in BV(\Omega, \{0,1\}). \tag{1.2}$$

Given $E \in BV_{\text{loc}}(\Omega, \{0,1\})$ we define its the *essential boundary* ∂E in Ω as

$$\partial E := \{x \in \Omega : \ 0 < |E \cap B_\rho(x)| < \omega_m \rho^m, \ \forall \rho > 0\}.$$

The set

$$\partial^* E := \left\{ x \in \partial E : \ \exists \nu_E(x) := -\lim_{r \to 0} \frac{\int_{B_r(x)} D\chi_E}{\int_{B_r(x)} |D\chi_E|}, \ |\nu_E(x)| = 1 \right\}$$

is called the *reduced boundary*. The vector ν_E is called the generalized outer unit normal to $\partial^* E$. Recall that [44, 63, 79] if $E \in BV(\Omega, \{0,1\})$, then

1) $\int_{\Omega \cap (\partial E \setminus \partial^* E)} |D\chi_E| = 0$;

2) $\overline{\partial^* E} \cap \Omega = \partial E \cap \Omega$;

3) $D\chi_E = \nu_E \mathcal{H}^{m-1} \llcorner \Omega \cap \partial^* E$ and $P(E, \Omega) = \mathcal{H}^{m-1}(\Omega \cap \partial^* E)$.

Remark 1.1. A) (Lower semicontinuity) From (1.1) it follows that the map $P(\cdot, \Omega)$ is $L^1_{\text{loc}}(\Omega)$-lower semicontinuous in $BV_{\text{loc}}(\Omega, \{0,1\})$:

$$E_j \overset{j \to +\infty}{\longrightarrow} E \text{ in } L^1_{\text{loc}}(\Omega) \ \Rightarrow \ P(E, V) \le \liminf_{j \to +\infty} P(E_j, V) \text{ for any } V \in \mathrm{Op}_b(\Omega).$$

B) (Compactness) Suppose that $\{E_j\} \in BV_{\text{loc}}(\Omega, \{0,1\})$ is such that

$$\sup_{j \ge 1} P(E_j, V) \le c_V < +\infty \qquad V \in \mathrm{Op}_b(\Omega).$$

Then there exists a subsequence $\{E_{j_k}\}$ and a set $E \in BV_{\text{loc}}(\Omega, \{0,1\})$ such that

$$E_{j_k} \to E \quad \text{in } L^1_{\text{loc}}(\Omega) \text{ as } k \to +\infty.$$

C) (Isoperimetric inequality) For any $E \in BV(\Omega, \{0,1\})$ one has

$$m\omega_m |E|^{\frac{n-1}{n}} \le P(E).$$

D) (Relative isoperimetric inequality) Suppose that Ω is a bounded open set with Lipschitz boundary. There exists $c(m,\Omega)\in(0,1)$ such that for any $E\in BV_{\rm loc}(\Omega,\{0,1\})$ one has

$$c(n,\Omega)\min\{|\Omega\cap E|^{\frac{n-1}{n}},|\Omega\cap E^c|^{\frac{n-1}{n}}\}\le P(E,\Omega).$$

Theorem 1.2 ([46])**.** *Let* $E\in BV_{\rm loc}(\mathbb{R}^m,\{0,1\})$*. Then for any* $x\in\partial^*E$

$$\lim_{\rho\to 0^+}\frac{|E\cap B_\rho(x)|}{|B_\rho(x)|}=\frac{1}{2},\qquad \lim_{\rho\to 0^+}\frac{P(E,B_\rho(x))}{\omega_{m-1}r^{m-1}}=1.$$

Theorem 1.3 ([11, Theorem 3.61])**.** *For every* $E\in BV_{\rm loc}(\mathbb{R}^m,\{0,1\})$

$$\mathcal{H}^{m-1}(\mathbb{R}^m\setminus(E^{(0)}\cup E\cup\partial^*E))=0.$$

Moreover, $\mathcal{H}^{m-1}(E^{(1/2)}\setminus\partial^*E)=0$.

1.2 Controlling the trace of a set by its perimeter

Let U be an open set with Lipschitz boundary and $E\in BV(\mathbb{R}^m)$. The function

$$\chi_E^+:\partial U\to\mathbb{R},\qquad \chi_E^+(x):=\lim_{r\to 0}\frac{|E\cap U\cap B_r(x)|}{|U\cap B_r(x)|}$$

is called the *interior trace of* E on ∂U. Analogously, the function

$$\chi_E^-:\partial U\to\mathbb{R},\qquad \chi_E^-(x):=\lim_{r\to 0}\frac{|E\cap U^c\cap B_r(x)|}{|U^c\cap B_r(x)|}$$

is called *exterior trace of* E on ∂U. The traces are well defined for $\mathcal{H}^{m-1}$-a.e. $x\in\partial U$ and $\chi_E^\pm\in L^1_{\rm loc}(\partial U)$ and we recall that $\chi_E^\pm\in L^1_{\rm loc}(\partial U)$. Moreover, the integration by parts formula holds [44]:

$$\int_U\chi_E\,{\rm div}\,g\,dx=-\int_U g\cdot D\chi_E+\int_{\partial U}(\chi_E^+-\chi_E^-)g\cdot\nu_U\,d\mathcal{H}^{m-1}\qquad\forall g\in C^1_c(\mathbb{R}^m,\mathbb{R}^m),\tag{1.3}$$

where ν_U is the outer unit normal to ∂U.

If $V\subseteq U$ is an open set with Lipschitz boundary, then

$$P(E,U)=P(E,V)+P(E,U\setminus\overline{V})+\int_{U\cap\partial V}|\chi_E^+-\chi_E^-|\,d\mathcal{H}^{m-1}.$$

The trace set of $E\subseteq U$ on ∂U is denoted by ${\rm Tr}(E)$; with a slight abuse of notation we set $\chi_{{\rm Tr}(E)}=\chi_E$. Note that

$$P(E,\overline{U}):=P(E,U)+\int_{\partial U}\chi_E\,d\mathcal{H}^{m-1}=P(E),\qquad E\subseteq U.$$

In general, even if $E\in BV(U,\{0,1\})$, the traces $\chi_E^\pm$ are in $L^1_{\rm loc}(\partial U)$, but not in $L^1(\partial U)$. For instance, if $U=\Big(\mathbb{R}\times(0,+\infty)\Big)\cup A\subset\mathbb{R}^2$ and $A=\bigcup_{m=2}^{+\infty}(m-\frac{1}{m^2},m+\frac{1}{m^2})\times(-1,0]$, then $E=A\in BV(U,\{0,1\})$, whereas $\mathcal{H}^1({\rm Tr}(E))=+\infty$. However, the following lemma shows that the $L^1(\partial U)$-norm of the trace of $E\in BV(U,\{0,1\})$ is controlled by $P(E,U)$ provided that U is a halfspace.

Lemma 1.4. *Suppose that U is a halfspace. Then for any $E \in BV(U,\{0,1\})$ and for any $\beta \in L^\infty(\partial U)$ one has*

$$\left|\int_{\partial U} \beta\, \chi_E \, d\mathcal{H}^{m-1}\right| \le \int_U |\beta \circ \pi|\, |D\chi_E| \le \|\beta\|_\infty\, P(E,U). \tag{1.4}$$

Here π is a projection of $\mathbb{R}^m$ onto $\partial\Omega \equiv \mathbb{R}^{m-1}$, i.e. $\pi(x',x_m) = x'$. In particular, $P(E) < +\infty$.

Proof. The last inequality of (1.4) is immediate. The first inequality is enough to be shown for $\beta \ge 0$ and $U := \{x_m > 0\}$.

If β is locally Lipschitz, define

$$\eta_k : [0,+\infty) \to [0,+\infty), \quad \eta_k(t) = \begin{cases} 1 & t \in [0,k], \\ k+1-x & t \in (k,k+1), \\ 0 & t \in [k+1,+\infty), \end{cases}$$

so that $\beta_k(x') := \eta_k(|x'|)\beta(x')$ is Lipschitz function of compact support. Choose a sequence $t_l \nearrow +\infty$ such that $\mathcal{H}^{m-1}(\partial^* E \cap \{x_m = t_l\}) = 0$ and

$$\lim_{l\to+\infty} \mathcal{H}^{m-1}(E \cap \{x_m = t_l\}) = 0. \tag{1.5}$$

Such sequence exists by [79, Proposition 2.16] and from the relation

$$|E| = \int_0^{+\infty} \mathcal{H}^{m-1}(E \cap \{x_m = t\})\, dt < +\infty.$$

Define $E_l := E \cap \{t_{l-1} < x_m < t_l\}$, $t_0 = 0$; clearly, $\mathrm{Tr}(E) = \mathrm{Tr}(E_1)$. Then as $\mathrm{div}((\beta_k\circ\pi)e_m) = 0$, from the divergence theorem in E_1 we obtain

$$0 = \int_{E_1} \mathrm{div}((\beta_k \circ \pi)e_m)\, dx = \int_{U\cap\partial^* E_1} (\beta_k \circ \pi)\, \nu_{E_1} \cdot e_m \, d\mathcal{H}^{m-1} - \int_{\partial U} \beta_k \chi_E \, d\mathcal{H}^{m-1}. \tag{1.6}$$

Whence the dominated convergence theorem and [79, Theorem 16.3] imply

$$\begin{aligned} \int_{\partial U} \beta\, \chi_E \, d\mathcal{H}^{m-1} &= \int_{U\cap\partial^* E_1} (\beta \circ \pi)\, \nu_{E_1} \cdot e_m \, d\mathcal{H}^{m-1} \\ &= \int_{\{0<x_m<t_1\}\cap\partial^* E} (\beta \circ \pi)\, \nu_E \cdot e_m \, d\mathcal{H}^{m-1} + \int_{\partial\{x_m<t_1\}} \beta \circ \pi\, \chi_E^+ \, d\mathcal{H}^{m-1}. \end{aligned} \tag{1.7}$$

Now if we set $U_1 := \{x_m > t_1\}$, $\beta_1 := \beta \circ \pi\Big|_{\partial U_1}$ and $\pi_1(x',x_m) = x'$ for $(x',x_m) \in U_1$, and using $\chi_E^+ = \chi_E^-$ on ∂U_1 and $\beta_1 \circ \pi = \beta \circ \pi$, as in (1.6) we get

$$\int_{\partial U_1} \beta_1 \chi_E^+ \, d\mathcal{H}^{m-1} = \int_{\{t_1<x_m<t_2\}\cap\partial^* E} (\beta \circ \pi)\, \nu_E \cdot e_m \, d\mathcal{H}^{m-1} + \int_{\partial\{x_m<t_2\}} \beta \circ \pi\, \chi_E^+ \, d\mathcal{H}^{m-1}.$$

But since $\int_{\partial\{x_m<t_1\}} \beta \circ \pi\, \chi_E^+ \, d\mathcal{H}^{m-1} = \int_{\partial U_1} \beta_1 \chi_E^+ \, d\mathcal{H}^{m-1}$, from (1.7) we obtain

$$\int_{\partial U} \beta\, \chi_E \, d\mathcal{H}^{m-1} = \int_{\{0<x_m<t_2\}\cap\partial^* E} (\beta \circ \pi)\, \nu_E \cdot e_m \, d\mathcal{H}^{m-1} + \int_{\partial\{x_m<t_2\}} \beta \circ \pi\, \chi_E^+ \, d\mathcal{H}^{m-1}.$$

By induction this extends to

$$\int_{\partial U} \beta\, \chi_E \, d\mathcal{H}^{m-1} = \int_{\{0<x_m<t_l\}\cap\partial^* E} \beta \circ \pi\, \nu_E \cdot e_m \, d\mathcal{H}^{m-1} + \int_{\partial\{x_m<t_l\}} \beta \circ \pi\, \chi_E^+ \, d\mathcal{H}^{m-1}.$$

Now by virtue of the monotone convergence theorem and (1.5) we can let $l \to +\infty$ to get

$$\int_{\partial U} \beta\, \chi_E \, d\mathcal{H}^{m-1} = \int_{\Omega\cap\partial^* E} \beta \circ \pi\, \nu_E \cdot e_m \, d\mathcal{H}^{m-1}.$$

Thus (1.4) follows. In particular, when $\beta \equiv 1,$ we have

$$P(E) = P(E,U) + \int_{\partial U} \chi_E d\mathcal{H}^{m-1} \leq 2P(E,U)$$

Notice that if $E \setminus U \neq \emptyset,$ then $\nu_E \cdot e_m < 1$ for some positive $\mathcal{H}^{m-1}$-measure subset of $\partial^* E$ (otherwise $\partial^* E$ would be hyperplane parallel to ∂U which would imply $|E| = +\infty$). Hence in this case

$$\int_{\partial U} \beta\, \chi_E \, d\mathcal{H}^{m-1} < \int_{\Omega\cap\partial^* E} \beta \circ \pi \, d\mathcal{H}^{m-1}. \tag{1.8}$$

This will be useful in proving, for example, Theorem 1.6.

If $\beta = \chi_{\hat{O}}$ for some open set $\hat{O} \subseteq \partial U,$ consider a sequence $\{\beta_k\}$ of nonnegative locally Lipschitz functions converging $\mathcal{H}^{m-1}$-a.e. to β on ∂U such that $\beta_k \leq \beta$ and $\operatorname{supp} \beta_k \subseteq \overline{\hat{O}}.$ By Fatou's lemma we get

$$\int_{\partial U} \beta\chi_E d\mathcal{H}^{m-1} \leq \liminf_{k\to+\infty} \int_{\partial U} \beta_k\chi_E d\mathcal{H}^{m-1} \leq \liminf_{k\to+\infty} \int_U \beta_k \circ \pi \, |D\chi_E| \leq \int_U \beta \circ \pi \, |D\chi_E|.$$

If $\beta = \chi_{\hat{A}}$ for some measurable $\hat{A} \subseteq \partial U,$ then the assertion follows from the regularity of measurable sets and from the previous observation. Finally, if $\beta \in L^\infty(\partial U)$ is any nonnegative function, then an approximation of β with a suitable sequence $\beta_k = \sum_{j=1}^m c_j \chi_{\hat{A}_j},$ where $c_j > 0$ and $\hat{A}_j \subseteq \partial U,$ $j = 1, \ldots, m$ are pairwise disjoint measurable sets, implies the statement. □

Remark 1.5. If U is a halfspace and $u \in BV(U),$ its trace belongs to $L^1(\partial U).$ Indeed, it is well-known that

$$\int_U |u| dx = \int_{-\infty}^0 \int_U \chi_{\{u<t\}}(x)\, dx dt + \int_0^{+\infty} \int_U \chi_{\{u>t\}}(x)\, dx dt, \tag{1.9}$$

$$\int_U |Du| = \int_{-\infty}^0 P(\{u<t\},U)\, dt + \int_0^{+\infty} P(\{u>t\},U)\, dt,$$

in particular, $\{u > t\}, \{u < s\} \in BV(U)$ for a.e. $t > 0$ and $s < 0.$ Using (1.4) with $\beta \equiv 1,$ for a.e. $t > 0$ and $s < 0$ we get

$$\int_{\partial U} \chi_{\{u>t\}} d\mathcal{H}^{m-1} \leq P(\{u>t\},U), \qquad \int_{\partial U} \chi_{\{u<s\}} d\mathcal{H}^{m-1} \leq P(\{u<s\},U)$$

and we obtain

$$\int_{\partial U} |u| \, d\mathcal{H}^{m-1} \leq \int_U |Du|.$$

Notice that for every $\beta \in L^\infty(\partial U)$ one has also

$$\int_{\partial U} \beta u \, d\mathcal{H}^{m-1} = -\int_{-\infty}^{0} \int_{\partial U} \beta \chi_{\{u<t\}} \, d\mathcal{H}^{m-1} dt + \int_0^{+\infty} \int_{\partial U} \beta \chi_{\{u>t\}} \, d\mathcal{H}^{m-1} dt. \tag{1.10}$$

As a corollary of Lemma 1.4 we get the comparison theorem of [10, page 216].

Theorem 1.6 (Comparison lemma)**.** *For any* $E \in BV(\mathbb{R}^m, \{0,1\})$ *and any closed convex set* $C \subseteq \mathbb{R}^m$ *the inequality* $P(E \cap C) \leq P(E)$ *holds; equality occurs if and only if* $|E \setminus C| = 0$.

Proof. As any convex set is an at most countable intersection of halfspaces, it is enough to prove the first assertion for the case when C is a halfspace. By Lemma 1.4 with $\beta \equiv 1$ and $U = C^\circ$ we have

$$\begin{aligned} P(E) =& P(E, C^\circ) + P(E, C^c) + \int_{\partial C} |\chi_E^+ - \chi_E^-| d\mathcal{H}^{m-1} \\ \geq& P(E \cap C) + \int_{\partial C} (\chi_E^+ - \chi_E^-) \, d\mathcal{H}^{m-1} + \int_{\partial C} |\chi_E^+ - \chi_E^-| d\mathcal{H}^{m-1} \geq P(E \cap C). \end{aligned} \tag{1.11}$$

To prove the second assertion we take any closed halfspace $H \supset C$; if $|E \cap H^c| > 0$, from (1.8) with $\beta \equiv 1$, (1.11) and the first assertion (applied with $E \cap H$) we get

$$P(E \cap C) = P(E) > P(E \cap H) \geq P(E \cap H \cap C) = P(E \cap C),$$

a contradiction. Hence, $|E \setminus H| = 0$, i.e. $|E \setminus C| = 0$. □

Another corollary of Lemma 1.4 is the following

Lemma 1.7. *Let* $U := \{x_m > 0\}$, $E \in BV(U, \{0,1\})$ *and* $H \subset \mathbb{R}^m$ *be the closed half-space whose outer unit normal* ν_H *satisfies* $\nu_H \cdot e_m \geq 0$. *Then*

$$P(E, U) \geq P(E \cap H, U). \tag{1.12}$$

Proof. Translating if necessary we may suppose that $0 \in \partial H \cap \partial U$. Note that if $\nu_H = e_m$ then (1.12) follows from Theorem 1.6. So we assume that $\nu_H \cdot e_m \in [0, 1)$. Let $(\partial U \cap \partial H)^\perp$ denote the 2-dimensional space orthogonal to $\partial U \cap \partial H$; clearly, ν_H and e_m form its basis. Take a unit vector $\nu_L \in (\partial U \cap \partial H)^\perp$ such that $\nu_L \cdot \nu_H = 0$ and $\nu_L \cdot e_m \leq 0$. Let $L \subset \mathbb{R}^m$ be the open subspace, whose outer unit normal is ν_L. Notice that by construction, traces of $E \cap L$ and $E \cap H$ on ∂U coincide, therefore

$$\begin{aligned} P(E, U) - P(E \cap H, U) = & P(E, U \cap L) + P(E, U \setminus \overline{L}) + \int_{U \cap \partial L} |\chi_{E \cap L} - \chi_{E \setminus L}| d\mathcal{H}^{m-1} \\ & - P(E \cap H, U) \geq P(E, U \cap L) + \int_{\partial U} \chi_{E \cap L} d\mathcal{H}^{m-1} - \left[P(E \cap H) + \int_{\partial U} \chi_{E \cap H} d\mathcal{H}^{m-1} \right] \\ =& P(E, L) - P(E \cap H). \end{aligned}$$

Hence, we need just to show

$$P(E,L) \geq P(E \cap H). \tag{1.13}$$

Since $E \cap H \subset L$ we have

$$P(E,L) = P(E,H^\circ) + P(E,L \setminus H) + \int_{U \cap \partial H} |\chi_{E \cap H} - \chi_{E \cap (L \setminus H)}| \, d\mathcal{H}^{m-1} \tag{1.14}$$

and

$$P(E \cap H) = P(E,H^\circ) + \int_{U \cap \partial H} \chi_{E \cap H} \, d\mathcal{H}^{m-1},$$

where H° is the interior of H. Applying Lemma 1.9 below with $U := \mathbb{R}^m \setminus H$, and $A = L \setminus H$ using also $E \subset U$ we get

$$P(E,L \setminus H) \geq \int_{\partial H} \chi_{E \cap (L \setminus H)} d\mathcal{H}^{m-1} = \int_{U \cap \partial H} \chi_{E \cap (L \setminus H)} d\mathcal{H}^{m-1}. \tag{1.15}$$

Now (1.13) follows from (1.14)-(1.15) and inequality $|a-b| \geq a-b$. □

Corollary 1.8. *Let E_0 be a closed convex set such that $\mathcal{H}^{m-1}$-a.e. $x \in \Omega \cap \partial E_0$ has an approximate outer unit normal $\nu_{E_0}(x)$ satisfying $\nu_{E_0}(x) \cdot e_m \geq 0$. Then $P(E_0,\Omega) \leq P(E,\Omega)$ for every $E \supseteq E_0$.*

Proof. Since E_0 is convex, we can choose countably many $\{x_j\} \subset \Omega \cap \partial^* E_0$, dense in $\Omega \cap \partial E_0$, such that $E_0 = \bigcap_{j \geq 1} H_{x_j}$, where H_{x_j} is the closed half space whose outer unit normal is $\nu_{E_0}(x_j)$. Then inductive application of Lemma 1.7 and the lower semicontinuity of perimeter imply $P(E_0,\Omega) \leq P(E,\Omega)$ for all $E \supseteq E_0$. □

The next lemma improves the assertion of Lemma 1.4 and is a localized version of [29, Lemma 4].

Lemma 1.9. *Assume that $\|\beta\|_\infty \leq 1$ and $E \in BV(\Omega,\{0,1\})$. Then for any open set $A \subseteq \Omega$ with $A \in BV_{\mathrm{loc}}(\mathbb{R}^m,\{0,1\})$ and*

$$\mathcal{H}^{m-1}\Big([\pi^{-1}(\pi(A)) \setminus A] \cap \Omega \cap \partial^* E\Big) = 0 \tag{1.16}$$

the inequality

$$P(E,A) - \int_{\partial\Omega} \beta \, \chi_{E \cap A} \, d\mathcal{H}^{m-1} \geq \frac{1 - \operatorname{ess\,sup} \beta}{2} \left[P(E,A) + \int_{\partial\Omega} \chi_{E \cap A} \, d\mathcal{H}^{m-1} \right] \tag{1.17}$$

holds.

Proof. Let us first show that if $F \subset \Omega$ has locally finite perimeter in $\mathbb{R}^m$, then

$$\chi_F \leq \chi_{\pi(F)} \quad \mathcal{H}^{m-1}\text{-a.e. on } \partial\Omega. \tag{1.18}$$

Set $\hat{G} := \{\hat{x} \in \mathrm{Tr}(F) : \chi_{\pi(F)}(\hat{x}) = 0\}$. For any $\epsilon > 0$ take an open set $\hat{O} \subseteq \partial\Omega$ such that $\hat{G} \subseteq \hat{O}$ and $\mathcal{H}^{m-1}(\hat{O} \setminus \hat{G}) < \epsilon$. Since $\mathcal{H}^{m-1}(\pi(F) \cap \hat{G}) = 0$, one has

$$
\begin{aligned}
|F \cap \pi^{-1}(\hat{G})| &= \int_{\pi^{-1}(\hat{G})} \chi_F \, dx = \int_0^{+\infty} dx_m \int_{\hat{G}} \chi_F(\hat{x}, x_m) d\mathcal{H}^{m-1}(\hat{x}) \\
&= \int_0^{+\infty} \mathcal{H}^{m-1}(\hat{G} \cap \{(\hat{x}, 0) : (\hat{x}, x_m) \in F\}) dx_m = \int_0^{+\infty} \mathcal{H}^{m-1}(\hat{G} \cap \pi(F)) dx_m = 0.
\end{aligned}
$$

Let $\hat{B}_\rho \subset \mathbb{R}^{m-1}$ denote the ball of radius $\rho > 0$ centered at the origin. Recall that for any $\gamma > 0$ the following estimate [63, page 35] holds:

$$
\int_{\hat{O} \cap \hat{B}_\rho} \chi_F d\mathcal{H}^{m-1} \leq P(F, (\hat{O} \cap \hat{B}_\rho) \times (0, \gamma)) + \frac{1}{\gamma} \int_{(\hat{O} \cap \hat{B}_\rho) \times (0,\gamma)} \chi_F \, dx.
$$

Then using $\hat{G} \subseteq \mathrm{Tr}(F)$, we establish

$$
\begin{aligned}
\mathcal{H}^{m-1}(\hat{G} \cap \hat{B}_\rho) \leq & \int_{\hat{O} \cap \hat{B}_\rho} \chi_F d\mathcal{H}^{m-1} \leq P(F, (\hat{O} \cap \hat{B}_\rho) \times (0, \gamma)) \\
& + \frac{1}{\gamma} \int_{(\hat{G} \cap \hat{B}_\rho) \times (0,\gamma)} \chi_F \, dx + \frac{1}{\gamma} \int_{((\hat{O} \setminus \hat{G}) \cap \hat{B}_\rho) \times (0,\gamma)} \chi_F \, dx \\
\leq & P(F, \hat{O} \times (0, \gamma)) + \frac{1}{\gamma} |F \cap \pi^{-1}(\hat{G})| + \mathcal{H}^{m-1}(\hat{O} \setminus \hat{G}) < P(F, \hat{O} \times (0, \gamma)) + \epsilon.
\end{aligned}
$$

Now letting $\epsilon, \gamma \to 0^+$ we get $\mathcal{H}^{m-1}(\hat{G} \cap \hat{B}_\rho) = 0$ and (1.18) follows from letting $\rho \to +\infty$.

We have

$$
\int_\Omega \chi_{\pi(A)} \circ \pi \, \frac{1 + \beta \circ \pi}{2} |D\chi_E| = \int_{\pi^{-1}(\pi(A))} \frac{1 + \beta \circ \pi}{2} |D\chi_E| = \int_A \frac{1 + \beta \circ \pi}{2} |D\chi_E|, \tag{1.19}
$$

where in the second equality we used (1.16). Moreover, from (1.18) with $F = A$ we get

$$
\int_{\partial\Omega} \frac{1 + \beta}{2} \chi_{E \cap A} \, d\mathcal{H}^{m-1} = \int_{\partial\Omega} \chi_A \, \frac{1 + \beta}{2} \chi_E \, d\mathcal{H}^{m-1} \leq \int_{\partial\Omega} \chi_{\pi(A)} \, \frac{1 + \beta}{2} \chi_E \, d\mathcal{H}^{m-1}. \tag{1.20}
$$

Now, using Lemma 1.4 with β replaced with $(1 + \beta)\chi_{\pi(A)}/2$, from (1.19) and (1.20) we obtain

$$
\int_{\partial\Omega} \frac{1 + \beta}{2} \chi_{E \cap A} \, d\mathcal{H}^{m-1} \leq \int_A \frac{1 + \beta \circ \pi}{2} |D\chi_E|. \tag{1.21}
$$

Finally, adding the identities

$$
\begin{aligned}
P(E, A) = \int_A |D\chi_E| = \int_A \frac{1 - \beta \circ \pi}{2} |D\chi_E| + \int_A \frac{1 + \beta \circ \pi}{2} |D\chi_E|, \\
- \int_{\partial\Omega} \beta \, \chi_{E \cap A} \, d\mathcal{H}^{m-1} = \int_{\partial\Omega} \frac{1 - \beta}{2} \chi_{E \cap A} \, d\mathcal{H}^{m-1} - \int_{\partial\Omega} \frac{1 + \beta}{2} \chi_{E \cap A} \, d\mathcal{H}^{m-1},
\end{aligned}
$$

and using (1.21) we deduce

$$
P(E, A) - \int_{\partial\Omega} \beta \, \chi_{E \cap A} \, d\mathcal{H}^{m-1} \geq \int_A \frac{1 - \beta \circ \pi}{2} |D\chi_E| + \int_{\partial\Omega} \frac{1 - \beta}{2} \chi_{E \cap A} \, d\mathcal{H}^{m-1}.
$$

This relation yields (1.17). $\square$

Chapter 2

Minimizers of anisotropic perimeters with cylindrical norms

The main results of this chapter is published in the Journal of Communications in Pure and Applied Analysis which is a joint work with G. Bellettini and M. Novaga [21]. We study various regularity properties of minimizers of the Φ–perimeter, where Φ is a norm. Under suitable assumptions on Φ and on the dimension of the ambient space, we prove that the boundary of a cartesian minimizer is locally a Lipschitz graph out of a closed singular set of small Hausdorff dimension. Moreover, we show the following anisotropic Bernstein-type result: any entire cartesian minimizer is the subgraph of a monotone function depending only on one variable.

In this chapter $\Omega \subseteq \mathbb{R}^{n+1}$ and $\widehat{\Omega} \subseteq \mathbb{R}^n$ are open sets. We often use the splitting $\mathbb{R}^{n+1} = \{(x,t) : \ x \in \mathbb{R}^n, \, t \in \mathbb{R}\}$ and write $\nu_E = (\widehat{\nu}_E, (\nu_E)_t)$ and $e_{n+1} = (0, \ldots, 0, 1)$. If $F \subseteq \mathbb{R}^{n+1}$, $x \in \mathbb{R}^n$, $\ t \in \mathbb{R}$ we let

$$F_t := \{y \in \mathbb{R}^n : \quad (y,t) \in F\}, \quad F_x := \{s \in \mathbb{R} : \quad (x,s) \in F\}. \tag{2.1}$$

Finally, for a function $u : \widehat{\Omega} \to \mathbb{R}$ we let

$$\mathrm{sg}(u) := \{(x,t) \in \mathbb{R}^{n+1} : \ x \in \widehat{\Omega}, \ u(x) > t\}$$

be the subgraph of u. We recall [63, 88] that

$$P_\Phi(\mathrm{sg}(u), \widehat{A} \times \mathbb{R}) < +\infty \qquad \forall \widehat{A} \in \mathrm{Op}_b(\widehat{\Omega}). \tag{2.2}$$

if and only if $u \in BV_{\mathrm{loc}}(\widehat{\Omega})$.

2.1 Sets of finite anisotropic perimeter

In this section we introduce the notion of anisotropic perimeter and comparison principles with convex sets, which will be useful in Chapter 3.

2.1.1 Norms

A norm on $\mathbb{R}^m$ is a convex function $\Psi : \mathbb{R}^m \to [0, +\infty)$ satisfying $\Psi(\lambda\xi) = |\lambda|\Psi(\xi)$ for all $\lambda \in \mathbb{R}$ and $\xi \in \mathbb{R}^m$, and for which there exists a constant $c > 0$ such that

$$c|\xi| \leq \Psi(\xi), \qquad \xi \in \mathbb{R}^m. \tag{2.3}$$

We let $B_\Psi := \{\xi \in \mathbb{R}^m : \Psi(\xi) \leq 1\}$, which is sometimes called Wulff shape, and $\Psi^o : (\mathbb{R}^m)^* \cong \mathbb{R}^m \to [0, +\infty)$ the dual norm of Ψ,

$$\Psi^o(\xi^*) = \sup\{\xi^* \cdot \xi : \ \xi \in B_\Psi\}, \qquad \xi^* \in \mathbb{R}^m,$$

where $(\mathbb{R}^m)^*$ is the dual of $\mathbb{R}^m$, which we identify with a copy of $\mathbb{R}^m$ itself, and $\cdot$ is the Euclidean scalar product. We have

$$\xi^* \cdot \xi \leq \Psi^o(\xi^*)\Psi(\xi), \quad \xi^* \in \mathbb{R}^m, \ \xi \in \mathbb{R}^m, \tag{2.4}$$

and $\Psi^{oo} = \Psi$. Unless otherwise specified, in this chapter we take $m \in \{n, n+1\}$. When $m = n+1$ we often split $\xi \in \mathbb{R}^{n+1}$ as $\xi = (\widehat{\xi}, \xi_{n+1}) \in \mathbb{R}^n \times \mathbb{R}$, and employ the symbol Φ (resp. φ) to denote a norm in $\mathbb{R}^{n+1}$ (resp. in $\mathbb{R}^n$). In $\mathbb{R}^{n+1}$ we frequently exploit the restriction $\Phi_{|_{\{\xi_{n+1}=0\}}}$ of Φ to the horizontal hyperplane $\{\xi_{n+1} = 0\}$, which is a norm on $\mathbb{R}^n$. Note that

$$\left(\Phi_{|_{\{\xi_{n+1}=0\}}}\right)^o \leq \Phi^o_{|_{\{\xi^*_{n+1}=0\}}}. \tag{2.5}$$

Indeed, let

$$\varphi := \Phi_{|_{\{\xi_{n+1}=0\}}} \quad \text{and} \quad \phi := \left(\Phi^o_{|_{\{\xi^*_{n+1}=0\}}}\right)^o.$$

Fix $\widehat{\xi}^* \in \mathbb{R}^n$ and choose $\widehat{\xi} \in \mathbb{R}^n$ such that $\varphi(\widehat{\xi}) = \Phi(\widehat{\xi}, 0) = 1$ and $\varphi^o(\widehat{\xi}^*) = \widehat{\xi} \cdot \widehat{\xi}^*$. Thus,

$$\varphi^o(\widehat{\xi}^*) = (\widehat{\xi}, 0) \cdot (\widehat{\xi}^*, 0) \leq \Phi^o(\widehat{\xi}^*, 0) = \phi^o(\widehat{\xi}^*).$$

Remark 2.1. Inequality (2.5) may be strict. For $\alpha \in (0, \pi/2)$ consider the symmetric parallelogram with vertices at $(1 \pm \cot\alpha, \pm 1)$, $(-1 \pm \cot\alpha, \pm 1)$, and let Φ_α be the Minkowski functional of P_α. Notice that

$$(\Phi_\alpha)_{|_{\{\xi_2=0\}}}(\xi_1) = |\xi_1|$$

and

$$(\Phi^o_\alpha)_{|_{\{\xi^*_2=0\}}}(1) = \Phi^o_\alpha(1, 0) = \sup\{\xi_1 : \ (\xi_1, \xi_2) \in P_\alpha\} = 1 + \cot\alpha,$$

thus

$$\left((\Phi_\alpha)_{|_{\{\xi_2=0\}}}\right)^o(1) = 1 < 1 + \cot\alpha = (\Phi^o_\alpha)_{|_{\xi^*_2=0}}(1).$$

In Lemma 2.20 we give necessary and sufficient conditions on Φ ensuring that equality in (2.5) holds.

Definition 2.2 (Cylindrical and conical norms). *We say that the norm $\Phi : \mathbb{R}^{n+1} \to [0, +\infty)$ is cylindrical over φ if*

$$\Phi(\widehat{\xi}, \xi_{n+1}) = \max\{\varphi(\widehat{\xi}), |\xi_{n+1}|\}, \qquad (\widehat{\xi}, \xi_{n+1}) \in \mathbb{R}^{n+1}, \tag{2.6}$$

where $\varphi : \mathbb{R}^n \to [0, +\infty)$ is a norm. We say that $\Phi : \mathbb{R}^{n+1} \to [0, +\infty)$ is conical over φ, if

$$\Phi(\xi) = \varphi(\widehat{\xi}) + |\xi_{n+1}|, \qquad (\widehat{\xi}, \xi_{n+1}) \in \mathbb{R}^{n+1}.$$

Notice that if Φ is cylindrical over φ then Φ^o is conical over φ^o, and vice-versa.

2.1.2 Anisotropic perimeters

Let $\Psi : \mathbb{R}^m \to [0, +\infty)$ be a norm and $O \subseteq \mathbb{R}^m$ be an open set. For any $E \in BV_{\mathrm{loc}}(O)$ and for any $A \in \mathrm{Op}_b(O)$ we define [9] the Ψ-perimeter of E in A as

$$P_\Psi(E, A) := \int_A \Psi^o(D\chi_E) = \sup\left\{ -\int_E \operatorname{div} \eta \, dx : \ \eta \in C^1_c(A, B_\Psi) \right\}.$$

It is known [9] that

$$P_\Psi(E, A) = \int_{A \cap \partial^* E} \Psi^o(\nu_E) \, d\mathcal{H}^{m-1}. \tag{2.7}$$

Remark 2.3. Given a norm $\Psi : \mathbb{R}^m \to [0, +\infty)$, and $E \in BV_{\mathrm{loc}}(\mathbb{R}^n, \{0, 1\})$ the map $\Omega \in \mathrm{Op}(\mathbb{R}^m) \mapsto P_\Psi(E, \Omega)$ extends to a Borel measure in $\mathbb{R}^m$, so that

$$P_\Psi(E, B) = \int_{B \cap \partial^* E} \Psi^o(\nu_E) d\mathcal{H}^{m-1}$$

for every Borel set $B \subseteq \mathbb{R}^m$.

Repeating the same procedure in the proof of [79, Theorem 16.3] one can show

Theorem 2.4. *If E and F are Caccioppoli sets, and we let*

$$\{\nu_E = \nu_F\} = \{x \in \partial^* E \cap \partial^* F : \ \nu_E(x) = \nu_F(x)\},$$

$$\{\nu_E = -\nu_F\} = \{x \in \partial^* E \cap \partial^* F : \ \nu_E(x) = -\nu_F(x)\},$$

then $E \cap F$, $E \setminus F$ and $E \cup F$ are locally finite perimeter sets with

$$\partial^*(E \cap F) \approx \big(F \cap \partial^* E\big) \cup \big(E \cap \partial^* F\big) \cup \big\{\nu_E = \nu_F\big\},$$

$$\partial^*(E \setminus F) \approx \big(F^{(0)} \cap \partial^* E\big) \cup \big(E \cap \partial^* F\big) \cup \big\{\nu_E = -\nu_F\big\},$$

$$\partial^*(E \cup F) \approx \big(F^{(0)} \cap \partial^* E\big) \cup \big(E^{(0)} \cap \partial^* F\big) \cup \big\{\nu_E = \nu_F\big\},$$

where $A \approx B$ means $\mathcal{H}^{m-1}(A \Delta B) = 0$. Moreover, for every Borel set $B \subseteq R^m$

$$P_\Psi(E \cap F, B) = P_\Psi(E, F \cap B) + P_\Psi(F, E \cap B) + \int_{\{\nu_E = \nu_F\} \cap B} \Psi^o(\nu_E) d\mathcal{H}^{m-1},$$

$$P_\Psi(E \setminus F, B) = P_\Psi(E, F^{(0)} \cap B) + P_\Psi(F, E \cap B) + \int_{\{\nu_E = -\nu_F\} \cap B} \Psi^o(\nu_E) d\mathcal{H}^{m-1},$$

$$P_\Psi(E \cup F, B) = P_\Psi(E, F^{(0)} \cap B) + P_\Psi(F, E^{(0)} \cap B) + \int_{\{\nu_E = \nu_F\} \cap B} \Psi^o(\nu_E) d\mathcal{H}^{m-1}.$$

Corollary 2.5. *For every $E, F \in BV_{\mathrm{loc}}(\mathbb{R}^m, \{0, 1\})$ and $O \in \mathrm{Op}(\mathbb{R}^m)$*

$$P_\Psi(E \cap F, O) + P_\Psi(E \cup F, O) \le P_\Psi(E, O) + P_\Psi(F, O).$$

Definition 2.6 (Minimizer of anisotropic perimeter). *We say that* $E \in BV_{\rm loc}(O, \{0,1\})$ *is a minimizer of* P_Ψ *by compact perturbations in* O *(briefly, a minimizer of* P_Ψ *in* O*) if*

$$P_\Psi(E, A) \leq P_\Psi(F, A) \tag{2.8}$$

for any $A \in {\rm Op}_b(O)$ *and* $F \in BV_{\rm loc}(O, \{0,1\})$ *such that* $E \Delta F \subset\subset A$.

From (2.7) it follows that if E is minimizer of P_Ψ in O, then so is $\mathbb{R}^m \setminus E$. If $m = 1$, then $\Phi(\xi) = \Phi(1)|\xi|$, thus $E \subset \mathbb{R}$ is a minimizer of P_Ψ in an open interval I if and only if it is a minimizer of the Euclidean perimeter, so E is of the form

$$\emptyset, \quad I, \quad (-\infty, \lambda) \cap I, \quad (\lambda, +\infty) \cap I, \qquad \lambda \in I. \tag{2.9}$$

The following example is based on a standard calibration argument[1].

Example 2.7 (Half-spaces). Let $H \subset \mathbb{R}^m$ be a half-space and $O \subseteq \mathbb{R}^m$ be open. Then $E = H \cap O$ is a minimizer of P_Ψ in O. Indeed, let $\zeta \in \mathbb{R}^m$ be such that $\Psi(\zeta) = 1$ and $\nu_H \cdot \zeta = \Psi^o(\nu_H)$. Consider $F \in BV_{\rm loc}(O, \{0,1\})$ with $E \Delta F \subset\subset A \subset\subset O$. Observe that $\partial^*(E \setminus F)$ can be written as a pairwise disjoint[2] union of $(\mathbb{R}^m \setminus F) \cap \partial E$, $E \cap \partial^* F$ and $J := \{z \in \partial E \cap \partial^* F : \nu_H(z) = -\nu_F(z)\}$ (see for example [79, Theorem 16.3]). For the vector field $N : \mathbb{R}^m \to \mathbb{R}^m$ constantly equal to ζ, we have

$$\begin{aligned} 0 &= \int_{E \setminus F} {\rm div}\, N \, dz \\ &= \int_{(\mathbb{R}^m \setminus F) \cap A \cap \partial E} \nu_H \cdot N \, d\mathcal{H}^{m-1} - \int_{E \cap A \cap \partial^* F} \nu_F \cdot N \, d\mathcal{H}^{m-1} \\ &+ \int_{J \cap A} \nu_H \cdot N \, d\mathcal{H}^{m-1} =: {\rm I} - {\rm II} + {\rm III}. \end{aligned} \tag{2.10}$$

Similarly,

$$\begin{aligned} 0 &= \int_{F \setminus E} {\rm div}(-N) \, dz \\ &= -\int_{(\mathbb{R}^m \setminus E) \cap A \cap \partial^* F} \nu_F \cdot N \, d\mathcal{H}^{m-1} + \int_{F \cap A \cap \partial E} \nu_H \cdot N \, d\mathcal{H}^{m-1} \\ &- \int_{J \cap A} \nu_F \cdot N \, d\mathcal{H}^{m-1} =: -{\rm IV} + {\rm V} - {\rm VI}. \end{aligned} \tag{2.11}$$

Adding (2.10)-(2.11) and using $\nu_F \cdot N \leq \Psi^o(\nu_F)$ we obtain

$$\begin{aligned} \int_{A \cap \partial E} \Psi^o(\nu_H) \, d\mathcal{H}^{m-1} &= \int_{A \cap \partial E} \nu_H \cdot N \, d\mathcal{H}^{m-1} = {\rm I} + {\rm III} + {\rm V} \\ = {\rm II} + {\rm IV} + {\rm VI} &= \int_{A \cap \partial^* F} \nu_F \cdot N \, d\mathcal{H}^{m-1} \leq \int_{A \cap \partial^* F} \Psi^o(\nu_F) \, d\mathcal{H}^{m-1}. \end{aligned}$$

[1] See for instance [1] for some definitions, results and references concerning calibrations.
[2] Up to sets of zero $\mathcal{H}^{m-1}$-measure.

The previous argument does not apply to a strip between two parallel planes.

Example 2.8 (Parallel planes). Let $n = 2$, and let $\Phi : \mathbb{R}^3 \to [0, +\infty)$ be cylindrical over the Euclidean norm. Given $a < b$ consider $E = \{(x,t) \in \mathbb{R}^3 : a < t < b\}$. Then E is not a minimizer of P_Φ in $\mathbb{R}^3$. Indeed, it is sufficient to compare E with the set $E \setminus C$, obtained from E by removing a sufficiently large cylinder $C = B_R \times [a,b]$ homothetic to B_Φ, where $B_R = \{x \in \mathbb{R}^2 : |x| < R\}$. Then $P_\Phi(E)$ is reduced by $2\pi R^2$ (the sum of the areas of the top and bottom facets of C), while it is increased by the lateral area $2\pi(b-a)R$ of C. Hence, for $R > 0$ sufficiently large, (2.8) is not satisfied. Notice that the horizontal sections of E are either empty or a plane, which both are minimizers of the Euclidean perimeter in $\mathbb{R}^2$.

The following proposition is the anisotropic analog of Comparison Theorem 1.6.

Theorem 2.9. *Let $E \in BV(\mathbb{R}^m, \{0,1\})$. Then $P_\Psi(E) \geq P_\Psi(E \cap H)$ for any closed convex set $C \subset \mathbb{R}^n$.*

Proof. It is enough to show the assertion when C is closed half space, since every convex closed set is the intersection of at most countably many closed halfspaces. Since Ψ^o is even, the following identities

$$P_\Psi(E) = P_\Psi(E, H^\circ) + P_\Psi(E, H^c) + \int_{\partial H} \Psi^o(\nu_H)|\chi_{E\cap H} - \chi_{E\cap H^c}| d\mathcal{H}^{m-1}$$

and

$$P_\Psi(E \cap H) = P_\Psi(E, H^\circ) + \int_{\partial H} \Psi^o(\nu_H)\chi_{E\cap H} d\mathcal{H}^{m-1}$$

are direct consequences of Theorem 2.4. Hence we just show $P_\Psi(E, H^c) \geq \int_{\partial H} \Psi^o(\nu_H)\chi_{E\cap H^c} d\mathcal{H}^{m-1}$ using a calibration argument as in Exercise 2.7. Let $\xi \in \mathbb{R}^m$ be such that $\xi \cdot \nu_H = \Psi^o(\nu_H)$ and $\Psi(\xi) = 1$. Then the vector field $\eta(x) \equiv \xi$ satisfies $\operatorname{div} \eta = 0$, and by Theorem 2.4

$$\begin{aligned} 0 &= \int_{E\cap H^c} \operatorname{div} \eta dx = \int_{\partial^*(E\cap H^c)} \eta \cdot \nu_{E\cap H^c} d\mathcal{H}^{m-1} \\ &= \int_{H^c \cap \partial^* E} \eta \cdot \nu_E d\mathcal{H}^{m-1} - \int_{E\cap\partial H} \eta \cdot \nu_H d\mathcal{H}^{m-1} - \int_{\{x \in \partial^* E \cap \partial H :\, \nu_E(x) = -\nu_H(x)\}} \eta \cdot \nu_H d\mathcal{H}^{m-1}. \end{aligned}$$

Therefore,

$$\begin{aligned} \int_{\partial H} \Psi^o(\nu_H)\chi_{E\cap H^c} d\mathcal{H}^{m-1} &\leq \int_{E\cap\partial H} \eta \cdot \nu_H d\mathcal{H}^{n-1} + \int_{\{x \in \partial^* E \cap \partial H :\, \nu_E(x) = -\nu_H(x)\}} \eta \cdot \nu_H d\mathcal{H}^{n-1} \\ &= \int_{H^c \cap \partial^* E} \eta \cdot \nu_E d\mathcal{H}^{n-1} \leq P_\Psi(E, H^c). \end{aligned}$$

$\square$

2.1.3 A Fubini-type theorem

Proposition 2.10. *Let $E \in BV_{\mathrm{loc}}(\Omega)$. Then for any $A \in \mathrm{Op}_b(\Omega)$*

$$\int_{A\cap\partial^* E} \Phi^o(\widehat{\nu}_E, 0)d\mathcal{H}^n = \int_{\mathbb{R}} dt \int_{A_t\cap\partial^* E_t} \Phi^o(\nu_{E_t}, 0)d\mathcal{H}^{n-1}, \tag{2.12}$$

$$\int_{A\cap\partial^* E} \Phi^o(0, (\nu_E)_t)d\mathcal{H}^n = \int_{\mathbb{R}^n} dx \int_{A_x\cap\partial^* E_x} \Phi^o(0,1)d\mathcal{H}^0. \tag{2.13}$$

where E_t and E_x are defined as (2.1), ν_{E_t} is a outer unit normal to $\partial^* E_t$ and ν_{E_x} is a outer unit normal to $\partial^* E_x$.

Proof. Let us prove (2.12). Notice that by [79, Theorem 18.11] for a.e. $t \in \mathbb{R}$

$$\mathcal{H}^{n-1}(\partial^* E_t \Delta (\partial^* E)_t) = 0, \quad \widehat{\nu}_E \neq 0, \quad \nu_{E_t} = \frac{\widehat{\nu}_E}{|\widehat{\nu}_E|}.$$

We can use the coarea formula [11, Theorem 2.93] with the function $f : \mathbb{R}^{n+1} \to \mathbb{R}$, $f(x,t) = t$. Then $\nabla f = e_{n+1}$ and its orthogonal projection $\nabla^E f$ on the approximate tangent space to $\partial^* E$ is $\nabla^E f = e_{n+1} - (e_{n+1}\cdot \nu_E)\nu_E$. Thus,

$$\begin{aligned}
&\int_{\mathbb{R}} dt \int_{A_t\cap\partial^* E_t} \Phi^o(\nu_{E_t}, 0)d\mathcal{H}^{n-1} = \int_{\mathbb{R}} dt \int_{(A\cap\partial^* E)\cap\{f=t\}} \Phi^o(\nu_{E_t}, 0)d\mathcal{H}^{n-1}\\
&= \int_{A\cap\partial^* E} \Phi^o(\nu_{E_t}, 0)|e_{n+1} - (e_{n+1}\cdot\nu_E)\nu_E|\, d\mathcal{H}^n\\
&= \int_{A\cap\partial^* E} \Phi^o(\nu_{E_t}, 0)\sqrt{1-|(\nu_E)_t|^2}\, d\mathcal{H}^n\\
&= \int_{A\cap\partial^* E} \Phi^o(\nu_{E_t}, 0)|\widehat{\nu}_E|\, d\mathcal{H}^n = \int_{A\cap\partial^* E} \Phi^o(\widehat{\nu}_E, 0)\, d\mathcal{H}^n.
\end{aligned}$$

Now, (2.13) follows from (2.7) and [88, Theorem 3.3]:

$$\begin{aligned}
\int_{A\cap\partial^* E} \Phi^o(0, (\nu_E)_t)d\mathcal{H}^n =& \Phi^o(0,1)\int_{A\cap\partial^* E} |(\nu_E)_t| d\mathcal{H}^n = \Phi^o(0,1)\int_A |D_t\chi_E|\\
=& \Phi^o(0,1)\int_{\mathbb{R}^n} dx \int_{A_x\cap\partial^* E_x} d\mathcal{H}^0.
\end{aligned}$$

□

Remark 2.11. Let $\Phi : \mathbb{R}^{n+1} \to [0, +\infty)$ be a norm. For notational simplicity set $\varphi_1 := \Phi_{|_{\{\xi_{n+1}=0\}}}$, $\varphi_2 := \Phi_{|_{\{\widehat{\xi}=0\}}}$. For $f \in BV_{\mathrm{loc}}(\Omega)$ and $A \in \mathrm{Op}_b(\Omega)$ we define

$$\int_A \varphi_1^o(D_x f) = \sup\Big\{\int_A f(x,t)\sum_{i=1}^n \frac{\partial\eta_i(x,t)}{\partial x_i}\, dxdt : \eta\in C_c^1(A; B_{\varphi_1})\Big\},$$

$$\int_A \varphi_2^o(D_t f) = \sup\Big\{\int_A f(x,s) D_t\eta(x,s)\, dxds : \eta \in C_c^1(A),\ \varphi_2(\eta)\le 1\Big\}.$$

With this notation (2.12) and (2.13) can be rewritten respectively as[3]

$$\int_A \varphi_1^o(D_x\chi_E) = \int_{\mathbb{R}} dt \int_{A_t} \varphi_1^o(D_x\chi_{E_t}),$$

$$\int_A \varphi_2^o(D_t\chi_E) = \int_{\mathbb{R}^n} dx \int_{A_x} \varphi_2^o(D_t\chi_{E_x}).$$

Remark 2.12. Suppose that $\Phi : \mathbb{R}^{n+1} \to [0,+\infty)$ is cylindrical over φ. Assume that $E \in BV_{\mathrm{loc}}(\Omega, \{0,1\})$ has the following property: for almost every $t \in \mathbb{R}$ the set E_t (horizontal section) is a minimizer of P_φ in Ω_t and for almost every $x \in \mathbb{R}^n$ the set E_x (vertical section) is a minimizer of Euclidean perimeter in Ω_x. Then by Remark 2.11 E is a minimizer of P_Φ in Ω.

Example 2.13. For any $l, \gamma \in \mathbb{R}$ we define the cones[4] in $\mathbb{R}^{n+1}$

$$C_1^{(n)}(l,\gamma) := (-\infty, l) \times \mathbb{R}^{n-1} \times (\gamma, +\infty),$$

$$C_2^{(n)}(l,\gamma) := (l, +\infty) \times \mathbb{R}^{n-1} \times (-\infty, \gamma).$$

From Example 2.7 and Remark 2.12 it follows that the following sets are minimizers of P_Φ in $\mathbb{R}^{n+1}$ provided that Φ satisfies (2.6):

a) $C_1^{(n)}(l_1,\gamma_1) \cup C_2^{(n)}(l_2,\gamma_2) \subset \mathbb{R}^{n+1}$, where $l_1 \le l_2,\ \ \gamma_1 \ge \gamma_2$ (see Figure 2.1).

b) The union of $C_1^{(n)}(l_1,\gamma)$ and the rotation of $C_2^{(n)}(l_2,\gamma)$ around the vertical axis x_{n+1} of α radiants (see Figure 2.2).

In general, a minimizer of P_Φ in Ω for a cylindrical Φ, need *not* satisfy the minimality property of horizontal sections in Remark 2.12.

Example 2.14 (Strips). Let $n = 2$, and let $\widehat{\Omega} = \mathbb{R} \times (0,\gamma) \subset \mathbb{R}^2$ with $\gamma > 0$. Take $\varphi^o(\xi_1^*, \xi_2^*) = |\xi_1^*| + |\xi_2^*|$, so that

$$P_\varphi(\widehat{F}, \widehat{A}) = \int_{\widehat{A}} |D_{x_1}\chi_{\widehat{F}}| + \int_{\widehat{A}} |D_{x_2}\chi_{\widehat{F}}|, \qquad \widehat{F} \in BV_{\mathrm{loc}}(\widehat{\Omega}, \{0,1\}),\ \widehat{A} \in \mathrm{Op}_b(\widehat{\Omega}).$$

We prove that if $l > \gamma > 0$ then the rectangle $\widehat{E} = (0,l) \times (0,\gamma)$ is a minimizer of P_φ in the strip $\widehat{\Omega}$. Let $\widehat{F} \in BV_{\mathrm{loc}}(\widehat{\Omega}, \{0,1\})$ be such that $\widehat{E}\Delta\widehat{F} \subset\subset \widehat{A} \subset\subset \widehat{\Omega}$. Let L_{x_1} stand for the vertical line passing through $(x_1, 0)$. If $\mathcal{H}^1(\widehat{F} \cap L_{x_1}) = 0$ or $\mathcal{H}^1(\widehat{F} \cap L_{x_1}) = \gamma$ for some $0 < x_1 < l$, then

$$P_\varphi(\widehat{F}, \widehat{A}) = P_\varphi(\widehat{F}, \widehat{A} \cap [(-\infty, x_1) \times (0,\gamma)]) + P_\varphi(\widehat{F}, \widehat{A} \cap [(x_1, +\infty) \times (0,\gamma)]).$$

[3] Following [88, Theorem 3.3] one can prove a more general statement, namely, if $f \in BV_{\mathrm{loc}}(A)$, then

$$\int_A \varphi_1^o(D_x f) = \int_{\mathbb{R}} dt \int_{A_t} \varphi_1^o(D_x(f|_{A_t})), \qquad \int_A \varphi_2^o(D_t f) = \int_{\mathbb{R}^n} dx \int_{A_x} \varphi_2^o(D_t(f|_{A_x})).$$

[4] A set $E \subseteq \mathbb{R}^m$ is a *cone* if there exists $x_0 \in \partial E$ such that for any $x \in E$ and $\lambda > 0$ it holds $x_0 + \lambda(x - x_0) \in E$.

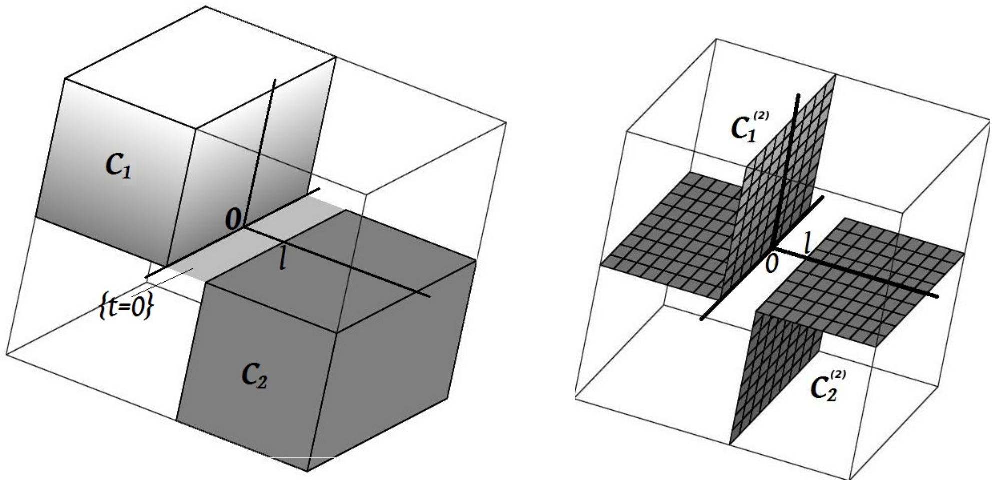

Figure 2.1: $C_1^{(2)}(0,0) \cup C_2^{(2)}(l,0)$ with $l > 0$ in Example 2.13(a) and its boundary. The picture below is a slight rotation of the picture above.

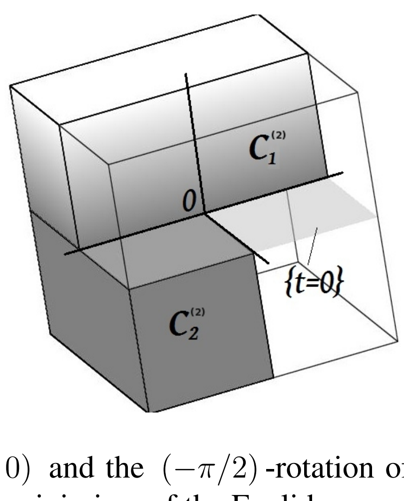

Figure 2.2: Union C of $C_1^{(2)}(0,0)$ and the $(-\pi/2)$-rotation of $C_2^{(2)}(0,0)$ in Example 2.13(b). Notice that C_t for $t=0$ is not a minimizer of the Euclidean perimeter in $\mathbb{R}^2$; however, this does not affect the minimality of C.

Hence

$$P_\varphi(\widehat{F}, \widehat{A} \cap [(-\infty, x_1) \times (0,\gamma)]) \geq P_\varphi(\widehat{E}, \widehat{A} \cap [(-\infty, x_1) \times (0,\gamma)]),$$

$$P_\varphi(\widehat{F}, \widehat{A} \cap [(x_1, +\infty) \times (0,\gamma)]) \geq P_\varphi(\widehat{E}, \widehat{A} \cap [(x_1, +\infty) \times (0,\gamma)]),$$

thus $P_\varphi(\widehat{F}, \widehat{A}) \geq P_\varphi(\widehat{E}, \widehat{A})$. Now assume that $0 < \mathcal{H}^1(\widehat{F} \cap L_{x_1}) < \gamma$ for all $x_1 \in (0,l)$. In this case $\int_{\widehat{A}} |D_{x_2}\chi_{\widehat{F}}| \geq 2l$. Indeed, since $\widehat{E}\Delta\widehat{F} \subset\subset \widehat{A}$, each vertical line L_{x_1}, $x_1 \in (0,l)$ should cross $\partial^*\widehat{F}$ at least twice. For a similar reason, taking into account the term $\int_{\widehat{A}} |D_{x_1}\chi_{\widehat{F}}|$ we may assume that $\widehat{E} \cap \partial^*\widehat{F}$ lies on two horizontal parallel lines at distance $\varepsilon \in (0,\gamma)$. Then by definition of φ-perimeter

$$P_\varphi(\widehat{F}, \widehat{A}) - P_\varphi(\widehat{E}, \widehat{A}) \geq 2l - 2\varepsilon \geq 2(l-\gamma) > 0.$$

This implies that $\widehat{E}$ is a minimizer of P_φ in $\widehat{\Omega}$. Notice that every horizontal section of $\widehat{E}$ is $(0,l)$, which is not a minimizer of the perimeter in $\mathbb{R}$.

Now, let $\Phi^o(\widehat{\xi^*}, \xi_3^*) = \varphi^o(\widehat{\xi^*}) + |\xi_3^*|$. By Proposition 2.21(b) below, $\widehat{E} \times \mathbb{R}$ is a minimizer of P_Φ in $\widehat{\Omega} \times \mathbb{R}$. Since Φ is symmetric with respect to relabelling the coordinate axis, the set $E = (0,l) \times \mathbb{R} \times (0,\gamma)$ is also a minimizer of P_Φ in $\mathbb{R} \times \mathbb{R} \times (0,\gamma)$. Notice that every horizontal section of E is a translation of the strip $(0,l) \times \mathbb{R}$, which is not a minimizer of P_φ in $\mathbb{R}^2$ according to Example 2.8.

Example 2.15. Let $\Phi^o(\xi_1^*, \xi_2^*) = |\xi_1^*| + |\xi_2^*|$. Given $l, \gamma \in \mathbb{R}$ suppose one of the following:

a) $l = 0$;

b) $l \geq 0 \geq \gamma$;

c) $l \geq \gamma > 0$.

Then the set $E = C_1^{(1)}(0,0) \cup C_2^{(1)}(l,\gamma)$ is a minimizer of P_Φ in $\mathbb{R}^2$ even though in case (c) for any $t \in (0,\gamma)$, the horizontal section E_t is not a minimizer of the perimeter in $\mathbb{R}$ (see (2.9)).

Indeed, if $l \geq 0 \geq \gamma$ then E satisfies the property in Remark 2.12. If $l = 0$ and $\gamma > 0$, then $\mathbb{R}^2 \setminus E$ is union of two disjoint cones satisfying property stated in Remark 2.12. Thus, in both cases E is a minimizer of P_Φ in $\mathbb{R}^2$.

Assume (c). By Remark 2.12 both $C_1 = (-\infty, 0) \times (0, +\infty)$ and $C_2 = (-\infty, \gamma) \times (l, +\infty)$ are minimizers of P_Φ in $\mathbb{R}^2$ (for brevity we do not write the dependence on l and γ). Consider arbitrary $F \in BV_{\mathrm{loc}}(\mathbb{R}^2, \{0,1\})$ with $E\Delta F \subset\subset (-M,M)^2$, for some $M > 0$.

If F perturbs the components C_1, C_2 of E separately, *i.e.* $F = F_1 \cup F_2$ and there exist disjoint open sets $A_1, A_2 \subset \mathbb{R}^2$ such that $C_i \Delta F_i \subset A_i$, $i = 1,2$, then by minimality of C_1, C_2 we have

$$\begin{aligned} P_\Phi(F, A_1 \cup A_2) =& P_\Phi(F_1, A_1) + P_\Phi(F_2, A_2) \geq P_\Phi(C_1, A_1) + P_\Phi(C_2, A_2) \\ =& P_\Phi(E, A_1 \cup A_2). \end{aligned}$$

On the other hand, it is not difficult to see that among all perturbations of E involving both components, the best one is obtained by inserting an horizontal strip as in Figure 2.3. However, because of the assumption $0 < \gamma \leq l$, this perturbation has larger Φ-perimeter than E. Consequently, E is a minimizer of P_Φ in $\mathbb{R}^2$.

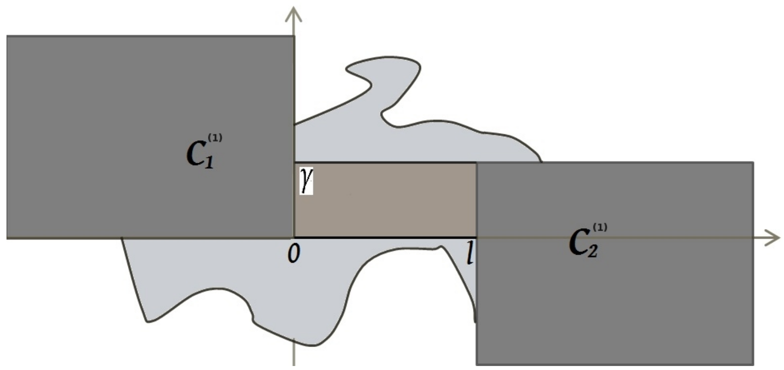

Figure 2.3: In case $0 < \gamma \leq l$, among all sets connecting two components of E the strip parallel to ξ_1-axis has the "smallest" Φ-perimeter.

2.2 Cylindrical minimizers

Let Φ be a norm on $\mathbb{R}^{n+1}$ and $\Omega = \widehat{\Omega} \times \mathbb{R}$.

Definition 2.16 (**Cylindrical minimizers**). *We say that a minimizer $E \subseteq \Omega$ of P_Φ in Ω is cylindrical over $\widehat{E}$ if $E = \widehat{E} \times \mathbb{R}$, where $\widehat{E} \subseteq \widehat{\Omega}$.*

The aim of this section is to characterize cylindrical minimizers of P_Φ. The idea here is that the (Euclidean) normal to the boundary of a cylindrical minimizer is horizontal, and therefore what matters, in the computation of the anisotropic perimeter, is only the horizontal section of the anisotropy. For this reason it is natural to introduce the following property, which informally requires the upper (and the lower) part of the boundary of the Wulff shape to be a generalized graph (hence possibly with vertical parts) over its projection on the horizontal hyperplane $\mathbb{R}^n \times \{0\}$.

Definition 2.17 (**Unit ball as a generalized graph in the vertical direction**). *We say that the boundary of the unit ball B_Φ of the norm $\Phi : \mathbb{R}^{n+1} \to [0, +\infty)$ is a generalized graph in the vertical direction if*

$$\Phi(\widehat{\xi}, \xi_{n+1}) \geq \Phi(\widehat{\xi}, 0), \qquad (\widehat{\xi}, \xi_{n+1}) \in \mathbb{R}^{n+1}. \tag{2.14}$$

Example 2.18. (a) If $\Phi(\widehat{\xi}, -\xi_{n+1}) = \Phi(\widehat{\xi}, \xi_{n+1})$ for all $(\widehat{\xi}, \xi_{n+1}) \in \mathbb{R}^{n+1}$, then ∂B_Φ is a generalized graph in the vertical direction. Indeed, from convexity

$$\Phi(\widehat{\xi}, 0) \leq \Phi\left(\frac{\widehat{\xi}}{2}, \frac{\xi_{n+1}}{2}\right) + \Phi\left(\frac{\widehat{\xi}}{2}, -\frac{\xi_{n+1}}{2}\right) = \Phi(\widehat{\xi}, \xi_{n+1}), \qquad (\widehat{\xi}, \xi_{n+1}) \in \mathbb{R}^{n+1}.$$

(b) There exists ∂B_Φ which is a generalized graph in the vertical direction, but Φ does not satisfy $\Phi(\widehat{\xi}, -\xi_{n+1}) = \Phi(\widehat{\xi}, \xi_{n+1})$. Fix some $\varepsilon \in (1/\sqrt{2}, \sqrt{2}]$ and consider the (symmetric convex) plane hexagon K_ε with vertices at $(1,0)$, $(\varepsilon, -\varepsilon)$, $(0,-1)$, $(-1,0)$, $(-\varepsilon, \varepsilon)$, $(0,1)$. Let $\Phi_\varepsilon : \mathbb{R}^2 \to [0,+\infty)$ be the Minkowski functional of K_ε. Then Φ_ε does not satisfy $\Phi_\varepsilon(\xi_1, -\xi_2) = \Phi_\varepsilon(\xi_1, \xi_2)$. But $\partial B_{\Phi_\varepsilon}$ is a generalized graph in the vertical direction.

Indeed, consider the straight line passing through $(1,0)$ and parallel to ξ_2 - axis. This line does not cross the interior of K_ε. Thus $\Phi_\varepsilon(1,\xi_2) \geq 1 = \Phi_\varepsilon(1,0)$. If $\xi_1 \neq 0$, then

$$\Phi_\varepsilon(\xi_1,\xi_2) = |\xi_1|\Phi_\varepsilon(1,\xi_2/\xi_1) \geq |\xi_1|\Phi_\varepsilon(1,0) = \Phi_\varepsilon(\xi_1,0).$$

If $\xi_1 = 0$, the inequality $\Phi_\varepsilon(\xi_1,\xi_2) \geq \Phi_\varepsilon(\xi_1,0)$ is obvious.

c) The norm $\Phi : \mathbb{R}^2 \to [0,+\infty)$, $\Phi(\xi_1,\xi_2) = \sqrt{\xi_1^2+\xi_1\xi_2+\xi_2^2}$ has a unit ball the boundary of which is not a generalized graph in the vertical direction, since $\Phi(2,0) = 2 > \sqrt{3} = \Phi(2,-1)$.

Lemma 2.19. *∂B_Φ is a generalized graph in the vertical direction if and only if ∂B_{Φ^o} is a generalized graph in the vertical direction.*

Proof. Suppose that ∂B_Φ is a generalized graph in the vertical direction. Let $\xi^* = (\widehat{\xi}^*, 0) \in \mathbb{R}^{n+1}$, and take $\xi = (\widehat{\xi}, \xi_{n+1}) \in \mathbb{R}^{n+1}$ such that $\Phi(\xi) = 1$ and

$$\widehat{\xi}\cdot\widehat{\xi}^* = \Phi^o(\widehat{\xi}^*,0) = (\widehat{\xi},\xi_{n+1})\cdot(\widehat{\xi}^*,0). \tag{2.15}$$

Since ∂B_Φ is a generalized graph in the vertical direction, we have $\Phi(\widehat{\xi},0) \leq \Phi(\xi) = 1$. Thus, by (2.4) and (2.15) we get $\Phi^o(\widehat{\xi}^*,0) \leq \Phi^o(\widehat{\xi}^*,\xi^*_{n+1})$, hence ∂B_{Φ^o} is a generalized graph in the vertical direction. The converse conclusion follows then from the equality $\Phi^{oo} = \Phi$. □

Lemma 2.20. *Equality in (2.5), namely*

$$\left(\Phi_{|_{\{\xi_{n+1}=0\}}}\right)^o = \Phi^o_{|_{\{\xi^*_{n+1}=0\}}}$$

holds if and only if ∂B_Φ is a generalized graph in the vertical direction.

Proof. Set $\varphi := \Phi_{|_{\{\xi_{n+1}=0\}}}$. Assume that ∂B_Φ is a generalized graph in the vertical direction. Let $(\widehat{\xi}^*,0) \in \mathbb{R}^{n+1}$ and take $\xi = (\widehat{\xi},\xi_{n+1}) \in \mathbb{R}^{n+1}$ such that $\Phi(\xi) = 1$ and (2.15) holds. By our assumption on ∂B_Φ it follows that $\varphi(\widehat{\xi}) = \Phi(\widehat{\xi},0) \leq \Phi(\widehat{\xi},\xi_{n+1}) = 1$, hence by (2.4)

$$\Phi^o(\widehat{\xi}^*,0) \leq \varphi^o(\widehat{\xi}^*)\varphi(\widehat{\xi}) \leq \varphi^o(\widehat{\xi}^*).$$

This and (2.5) imply $\varphi^o(\widehat{\xi}^*) = \Phi^o(\widehat{\xi}^*,0)$, *i.e.* $\left(\Phi_{|_{\{\xi_{n+1}=0\}}}\right)^o = \Phi^o_{|_{\{\xi^*_{n+1}=0\}}}$.

Now assume that equality in (2.5) holds. Take any $\xi = (\widehat{\xi},\xi_{n+1}) \in \mathbb{R}^{n+1}$ and select $\widehat{\xi}^* \in \mathbb{R}^n$ such that $\varphi^o(\widehat{\xi}^*) = \Phi^o(\widehat{\xi}^*,0) = 1$ and $\varphi(\widehat{\xi}) = \widehat{\xi}\cdot\widehat{\xi}^*$. Then by (2.4)

$$\Phi(\widehat{\xi},0) = \varphi(\widehat{\xi}) = \widehat{\xi}\cdot\widehat{\xi}^* = (\widehat{\xi},\xi_{n+1})\cdot(\widehat{\xi}^*,0) \leq \Phi(\widehat{\xi},\xi_{n+1}).$$

□

Proposition 2.21 (Cylindrical minimizers). *Let $\Phi : \mathbb{R}^{n+1} \to [0,+\infty)$ be a norm and let $\widehat{E} \in BV_{\mathrm{loc}}(\widehat{\Omega})$. The following assertions hold.*

(a) If $\widehat{E}\times\mathbb{R}$ is a minimizer of P_Φ in $\widehat{\Omega}\times\mathbb{R}$, then $\widehat{E}$ is a minimizer of P_ϕ in $\widehat{\Omega}$, where

$$\phi := \left(\Phi^o_{|_{\{\xi^*_{n+1}=0\}}}\right)^o.$$

(b) If ∂B_Φ is a generalized graph in the vertical direction and $\widehat{E}$ is a minimizer of P_φ in $\widehat{\Omega}$, where $\varphi := \Phi_{|_{\{\xi_{n+1}=0\}}}$, then $\widehat{E}\times\mathbb{R}$ is a minimizer of P_Φ in $\widehat{\Omega}\times\mathbb{R}$.

Remark 2.22. In general $\phi\neq\varphi$ (see Remark 2.1 and Lemma 2.20).

Proof. (a) Take $\widehat{A}\in \mathrm{Op}_b(\widehat{\Omega})$, $\widehat{F}\in BV_{\mathrm{loc}}(\widehat{\Omega})$ with $\widehat{E}\Delta\widehat{F}\subset\subset\widehat{A}$. For any $m>0$ set $I_m := (-m,m)$, and define

$$F_m := [E\setminus(\mathbb{R}^n\times I_m)]\cup[\widehat{F}\times I_m],$$

where $E=\widehat{E}\times\mathbb{R}$. Then $E\Delta F_m\subset\subset \widehat{A}\times I_{m+1}\subset\subset\widehat{\Omega}\times\mathbb{R}$ and, by minimality,

$$P_\Phi(E,\widehat{A}\times I_{m+1})\le P_\Phi(F_m,\widehat{A}\times I_{m+1}). \tag{2.16}$$

Writing $\nu_E=(\widehat{\nu_E},(\nu_E)_t)$, we have $\nu_E=(\nu_{\widehat{E}},0)$ $\mathcal{H}^n$-almost everywhere on ∂^*E. Hence

$$\begin{aligned} P_\Phi\left(E,\widehat{A}\times I_{m+1}\right) =& \int_{[\widehat{A}\times I_{m+1}]\cap\partial^*E}\Phi^o\left(\widehat{\nu_E},(\nu_E)_t\right)\,d\mathcal{H}^n\\ =&\int_{[\widehat{A}\times I_{m+1}]\cap\partial^*E}\phi^o(\nu_{\widehat{E}})\,d\mathcal{H}^n\\ =&2(m+1)\int_{\widehat{A}\cap\partial^*\widehat{E}}\phi^o(\nu_{\widehat{E}})\,d\mathcal{H}^{n-1}=2(m+1)P_\phi(\widehat{E},\widehat{A}). \end{aligned}\tag{2.17}$$

Similarly, $\nu_{F_m}=(\nu_{\widehat{F}},0)$ on $(\partial^*\widehat{F})\times I_m$, $\nu_{F_m}=(0,\pm1)$ on $(\widehat{E}\Delta\widehat{F})\times\{\pm m\}$ and $\nu_{F_m}=(\nu_{\widehat{E}},0)$ on $(\partial^*\widehat{F})\times(I_{m+1}\setminus\overline{I_m})$. As a consequence,

$$\begin{aligned} P_\Phi(F_m,\widehat{A}\times I_{m+1}) =& \int_{[\widehat{A}\times I_m]\cap\partial^*F_m}\Phi^o(\nu_{F_m})\,d\mathcal{H}^n\\ &+\int_{[\widehat{A}\times\{\pm m\}]\cap\partial^*F_m}\Phi^o(\nu_{F_m})\,d\mathcal{H}^n\\ &+\int_{[\widehat{A}\times(I_{m+1}\setminus\overline{I_m})]\cap\partial^*F_m}\Phi^o(\nu_{F_m})\,d\mathcal{H}^n\\ =&2mP_\phi(\widehat{F},\widehat{A})+2\Phi^o(0,1)\mathcal{H}^n(\widehat{E}\Delta\widehat{F})+2P_\phi(\widehat{E},\widehat{A}). \end{aligned}\tag{2.18}$$

From (2.17), (2.18) and (2.16), it follows

$$P_\phi(\widehat{E},\widehat{A})\le P_\phi(\widehat{F},\widehat{A})+\frac{\Phi^o(0,1)}{m}\mathcal{H}^n(\widehat{F}\Delta\widehat{E}).$$

Letting $m\to+\infty$ we get $P_\phi(\widehat{E},\widehat{A})\le P_\phi(\widehat{F},\widehat{A})$, and assertion (a) follows.

(b) By Lemma 2.20, $\varphi^o = \Phi^o_{|\{\xi^*_{n+1}=0\}}$. Take $F \in BV_{\mathrm{loc}}(\widehat{\Omega} \times \mathbb{R})$, and let $\widehat{A} \in \mathrm{Op}_b(\widehat{\Omega})$ and $M > 0$ be such that $E \Delta F \subset\subset \widehat{A} \times I_M$, where $I_M := (-M, M)$. Then $\widehat{E} \Delta F_t \subset\subset \widehat{A}$ for all $t \in (-M, M)$ and since $\widehat{E}$ is a minimizer of P_φ in $\widehat{\Omega}$, using (2.14) and (2.12) we get

$$\begin{aligned} P_\Phi(F, \widehat{A} \times I_M) &= \int_{(\widehat{A} \times I_M) \cap \partial^* F} \Phi^o(\widehat{\nu_F}, (\nu_F)_t) \, d\mathcal{H}^n \geq \int_{(\widehat{A} \times I_M) \cap \partial^* F} \Phi^o(\widehat{\nu_F}, 0) \, d\mathcal{H}^n \\ &= \int_{-M}^{M} P_\varphi(F_t, \widehat{A}) \, dt \geq \int_{-M}^{M} P_\varphi(\widehat{E}, \widehat{A}) \, dt = P_\Phi(E, \widehat{A} \times I_M), \end{aligned}$$

and assertion (b) follows. □

Example 2.23 (Characterization of cylindrical minimizers for a cubic anisotropy). Proposition 2.21 allows us to classify the cylindrical minimizers of P_Φ for suitable choices of the dimension and of the anisotropy. Take $n = 2$, $\widehat{\Omega} = \mathbb{R}^2$, and let

$$B_\Phi = [-1, 1]^3;$$

in particular, ∂B_Φ is a generalized graph in the vertical direction and B_φ is the square $[-1, 1]^2$ in the (horizontal) plane. The minimizers of P_φ are classified as follows [94, Theorems 3.8 (ii) and 3.11 (2)]: the infinite cross $\widehat{C} = \{|x_1| > |x_2|\}$ and its complement, the subgraphs and epigraphs $\widehat{S}$ of monotone functions of one variable, and suitable unions $\widehat{U}$ of two connected components, each of which is the subgraph of a monotone function of one variable. Then Proposition 2.21 (b) implies that

$$\widehat{C} \times \mathbb{R}, \quad (\mathbb{R}^2 \setminus \widehat{C}) \times \mathbb{R}, \quad \widehat{S} \times \mathbb{R}, \quad \widehat{U} \times \mathbb{R}$$

are the *only* cylindrical minimizers of P_Φ in $\mathbb{R}^3$. The same result holds if B_φ is a parallelogram centered at the origin, and ∂B_Φ is any generalized graph in the vertical direction such that $B_\varphi = B_\Phi \cap \{\xi_3 = 0\}$.

2.3 Cartesian minimizers for partially monotone norms

Let $\Phi : \mathbb{R}^{n+1} \to [0, +\infty)$ be a norm.

Definition 2.24 (Cartesian minimizers). *We call a minimizer $E \subseteq \Omega = \widehat{\Omega} \times \mathbb{R}$ a cartesian minimizer of P_Φ in $\Omega = \widehat{\Omega} \times \mathbb{R}$ if $E = \mathrm{sg}(u)$ for some function $u : \widehat{\Omega} \to \mathbb{R}$.*

Let $v \in BV_{\mathrm{loc}}(\widehat{\Omega})$; in what follows the symbol $\int_{\widehat{A}} \Phi^o(-Dv, 1)$ means

$$\begin{aligned} \int_{\widehat{A}} \Phi^o(-Dv, 1) = \sup \Big\{ \int_{\widehat{A}} \Big(v \sum_{j=1}^{n} \frac{\partial \eta_j}{\partial x_j} + \eta_{n+1} \Big) \, dx : \\ \eta = (\eta_1, \dots, \eta_{n+1}) \in C^1_c(\widehat{A}, B_\Phi) \Big\}. \end{aligned}$$

If $v \in W^{1,1}_{\mathrm{loc}}(\widehat{\Omega})$ we have [11, Theorem 2.91]

$$\begin{aligned} P_\Phi(\mathrm{sg}(v), \widehat{A} \times \mathbb{R}) &= \int_{(\widehat{A}\times\mathbb{R})\cap\partial\mathrm{sg}(v)} \Phi^o(\nu_{\mathrm{sg}(v)})\, d\mathcal{H}^n \\ &= \int_{(\widehat{A}\times\mathbb{R})\cap\partial\mathrm{sg}(v)} \Phi^o(-\nabla v, 1)\, \frac{d\mathcal{H}^n}{\sqrt{1+|\nabla v|^2}} = \int_{\widehat{A}} \Phi^o(-\nabla v, 1)\, dx. \end{aligned}$$

Using the techniques in [42], the previous equality extends to any $v \in BV_{\mathrm{loc}}(\widehat{\Omega})$:

$$P_\Phi(\mathrm{sg}(v), \widehat{A} \times \mathbb{R}) = \int_{(\widehat{A}\times\mathbb{R})\cap\partial^*\mathrm{sg}(v)} \Phi^o(\nu_{\mathrm{sg}(v)})\, d\mathcal{H}^n = \int_{\widehat{A}} \Phi^o(-Dv, 1). \tag{2.19}$$

Accordingly, we define the functional $\mathcal{G}_{\Phi^o} : BV_{\mathrm{loc}}(\widehat{\Omega}) \times \mathrm{Op}_b(\widehat{\Omega}) \to [0, +\infty)$ as follows:

$$\mathcal{G}_{\Phi^o}(v, \widehat{A}) := \int_{\widehat{A}} \Phi^o(-Dv, 1), \qquad v \in BV_{\mathrm{loc}}(\widehat{\Omega}),\ \widehat{A} \in \mathrm{Op}_b(\widehat{\Omega}).$$

Definition 2.25. *We say that $u \in BV_{\mathrm{loc}}(\widehat{\Omega})$ is a minimizer of $\mathcal{G}_{\Phi^o}$ by compact perturbations in $\widehat{\Omega}$ (briefly, a minimizer of $\mathcal{G}_{\Phi^o}$ in $\widehat{\Omega}$), and we write*

$$u \in \mathcal{M}_{\Phi^o}(\widehat{\Omega}),$$

if for any $\widehat{A} \in \mathrm{Op}_b(\widehat{\Omega})$ and $v \in BV_{\mathrm{loc}}(\widehat{\Omega})$ with $\mathrm{supp}\,(u - v) \subset\subset \widehat{A}$ one has

$$\mathcal{G}_{\Phi^o}(u, \widehat{A}) \le \mathcal{G}_{\Phi^o}(v, \widehat{A}).$$

Note that $\mathcal{M}_{\Phi^o}(\widehat{\Omega}) \neq \emptyset$ since linear functions on $\widehat{\Omega}$ belong[5] to $\mathcal{M}_{\Phi^o}(\widehat{\Omega})$. Observe also that if $u \in \mathcal{M}_{\Phi^o}(\widehat{\Omega})$ then $u + c \in \mathcal{M}_{\Phi^o}(\widehat{\Omega})$ for any $c \in \mathbb{R}$.

We shall need the following standard result.

Theorem 2.26 (Compactness). *Let $\Phi : \mathbb{R}^{n+1} \to [0, +\infty)$ be a norm. If $u_k \in \mathcal{M}_{\Phi^o}(\widehat{\Omega})$, $u \in L^1_{\mathrm{loc}}(\widehat{\Omega})$ and $u_k \to u$ in $L^1_{\mathrm{loc}}(\widehat{\Omega})$ as $k \to +\infty$, then $u \in \mathcal{M}_{\Phi^o}(\widehat{\Omega})$.*

Proof. The proof is the same as in [93, Theorem 3.4] making use of lower semicontinuity of P_Φ, (2.3) and the inequality $\Phi^o(-Dw, 1) \le \Phi^o(Dw, 0) + \Phi^o(0, 1)$. □

The aim of this section is to show the relations between minimizers and cartesian minimizers, under a special assumption on the norm.

Definition 2.27 (Partially monotone norm). *The norm $\Phi : \mathbb{R}^{n+1} \to [0, +\infty)$ is called partially monotone if given $\xi = (\widehat{\xi}, \xi_{n+1}) \in \mathbb{R}^{n+1}$ and $\eta = (\widehat{\eta}, \eta_{n+1}) \in \mathbb{R}^{n+1}$ we have*

$$\Phi(\widehat{\xi}, 0) \le \Phi(\widehat{\eta}, 0), \;\; \Phi(0, \xi_{n+1}) \le \Phi(0, \eta_{n+1}) \quad \Longrightarrow \quad \Phi(\xi) \le \Phi(\eta). \tag{2.20}$$

[5] If u is linear, then $\mathrm{sg}(u)$ is the intersection of a half-space with $\widehat{\Omega} \times \mathbb{R}$, hence $\mathrm{sg}(u)$ is a minimizer of P_Φ in $\widehat{\Omega} \times \mathbb{R}$ (Example 2.7) and $u \in \mathcal{M}_{\Phi^o}(\widehat{\Omega})$ (see Theorem 2.32(a) below).

Example 2.28. The following norms on $\mathbb{R}^{n+1}$ are partially monotone: $\Phi(\widehat{\xi}, \xi_{n+1}) = \max\{\varphi(\widehat{\xi}), |\xi_{n+1}|\}$; $\Phi(\widehat{\xi}, \xi_{n+1}) = ([\varphi(\widehat{\xi})]^p + |\xi_{n+1}|^p)^{1/p}$, where $\varphi : \mathbb{R}^n \to [0, +\infty)$ is a norm and $p \in [1, +\infty)$.

Proposition 2.29 (Characterization of partially monotone norms). *The norm $\Phi : \mathbb{R}^{n+1} \to [0, +\infty)$ is partially monotone if and only if there exists a positively one-homogeneous convex function $\omega : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ satisfying*

$$\omega(1,0), \omega(0,1) > 0, \ \omega(s_1, s_2) \le \omega(t_1, t_2), \quad 0 \le s_i \le t_i, \ i = 1, 2, \tag{2.21}$$

such that

$$\Phi(\widehat{\xi}, \xi_{n+1}) = \omega(\varphi(\widehat{\xi}), |\xi_{n+1}|), \tag{2.22}$$

where $\varphi = \Phi_{|\{\xi_{n+1}=0\}}$.

Proof. Concerning the "if" part, one checks that the function Φ defined as (2.22) is a partially monotone norm on $\mathbb{R}^{n+1}$. Now, let us prove the "only if" part. Choose any $\widehat{\eta} \in \mathbb{R}^n$ with $\varphi(\widehat{\eta}) = 1$ and define the function $\omega : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ as $\omega(s,t) := \Phi(s\widehat{\eta}, t)$ for $s, t \ge 0$. Since Φ is convex and positively one-homogeneous, so is ω. Moreover, the relations $\Phi(\widehat{\eta}, 0) = 1$, $\Phi(0,1) > 0$ and partial monotonicity of Φ imply that ω satisfies (2.21). Now it remains to prove (2.22). Comparing $\xi = (\widehat{\xi}, \xi_{n+1}) \in \mathbb{R}^{n+1}$ with $\eta = (\varphi(\widehat{\xi})\widehat{\eta}, |\xi_{n+1}|) \in \mathbb{R}^{n+1}$ in (2.20) and using the relation $\Phi(0, \xi_{n+1}) = \Phi(0, |\xi_{n+1}|)$ and partial monotonicity, we find

$$\Phi(\widehat{\xi}, \xi_{n+1}) = \Phi(\varphi(\widehat{\xi})\widehat{\eta}, |\xi_{n+1}|) = \omega(\varphi(\widehat{\xi}), |\xi_{n+1}|).$$

□

Notice that for Φ as in (2.22) we have

$$\Phi^o(\widehat{\xi^*}, \xi^*_{n+1}) = \omega^o(\varphi^o(\widehat{\xi^*}), |\xi^*_{n+1}|), \tag{2.23}$$

where $\omega^o : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ is defined as

$$\begin{aligned} \omega^o(s_1^*, s_2^*) = \sup\{s_1 s_1^* + s_2 s_2^* : s_1, s_2 \in [0, +\infty), \\ \omega(s_1, s_2) \le 1\}, \ s_1^*, s_2^* \in [0, +\infty). \end{aligned} \tag{2.24}$$

Indeed, take any $\xi^* = (\widehat{\xi^*}, \xi^*_{n+1}) \in \mathbb{R}^{n+1}$. Let $\xi = (\widehat{\xi}, \xi_{n+1}) \in \mathbb{R}^{n+1}$ be such that $\Phi(\xi) = \omega(\varphi(\widehat{\xi}), |\xi_{n+1}|) \le 1$ and $\xi \cdot \xi^* = \Phi^o(\xi^*)$. Then using (2.4) twice we get

$$\begin{aligned} \Phi^o(\xi^*) =& \widehat{\xi} \cdot \widehat{\xi^*} + \xi_{n+1} \cdot \xi^*_{n+1} \le \varphi(\widehat{\xi})\varphi^o(\widehat{\xi^*}) + |\xi_{n+1}| \cdot |\xi^*_{n+1}| \\ \le& \omega(\varphi(\widehat{\xi}), |\xi_{n+1}|)\omega^o(\varphi^o(\widehat{\xi^*}), |\xi^*_{n+1}|) \le \omega^o(\varphi^o(\widehat{\xi^*}), |\xi^*_{n+1}|). \end{aligned} \tag{2.25}$$

On the other hand, for any $\xi^* \in \mathbb{R}^{n+1}$ there exist $\widehat{\xi} \in \mathbb{R}^n$ such that $\varphi(\widehat{\xi}) \le 1$ and

$$\widehat{\xi} \cdot \widehat{\xi^*} = \varphi^o(\widehat{\xi^*}).$$

Moreover, by definition of ω^o one can find $(s_1, s_2) \in [0, +\infty) \times [0, +\infty)$ such that $\omega(s_1, s_2) \leq 1$ and $\omega^o(\varphi^o(\widehat{\xi^*}), |\xi^*_{n+1}|) = s_1\varphi^o(\widehat{\xi^*}) + s_2|\xi^*_{n+1}|$. Using (2.21) for $(s_1\varphi(\widehat{\xi}), s_2 \operatorname{sign}(\xi^*_{n+1}))$ and (s_1, s_2) one has

$$\Phi(s_1\widehat{\xi}, s_2 \operatorname{sign}(\xi^*_{n+1})) = \omega(s_1\varphi(\widehat{\xi}), s_2|\operatorname{sign}(\xi_{n+1})|) \leq \omega(s_1, s_2) \leq 1.$$

Thus,

$$\begin{aligned} \omega^o(\varphi^o(\widehat{\xi^*}), |\xi^*_{n+1}|) =& s_1\varphi^o(\widehat{\xi^*}) + s_2|\xi^*_{n+1}| = (s_1\widehat{\xi}) \cdot \widehat{\xi^*} + (s_2 \operatorname{sign}(\xi^*_{n+1})) \cdot \xi^*_{n+1} \\ &\leq \Phi(s_1\widehat{\xi}, s_2 \operatorname{sign}(\xi^*_{n+1}))\Phi^o(\widehat{\xi^*}, \xi^*_{n+1}) \leq \Phi^o(\widehat{\xi^*}, \xi^*_{n+1}). \end{aligned} \tag{2.26}$$

From (2.25)-(2.26) we get (2.23).

Remark 2.30. The norm $\Phi : \mathbb{R}^{n+1} \to [0, +\infty)$ is partially monotone if and only if Φ^o is partially monotone.

We give the proof of the following lemma which is used in the proof of Theorem 2.32.

Lemma 2.31. *Suppose that $\Phi : \mathbb{R}^{n+1} \to [0, +\infty)$ is a partially monotone norm, $E, F \in BV_{\text{loc}}(\widehat{\Omega} \times \mathbb{R})$ such that for every $\widehat{A} \in \operatorname{Op}_b(\widehat{\Omega})$*

$$\begin{aligned} \int_{\widehat{A}\times\mathbb{R}} \Phi^o(D_x\chi_E, 0) &\leq \int_{\widehat{A}\times\mathbb{R}} \Phi^o(D_x\chi_F, 0), \\ \int_{\widehat{A}\times\mathbb{R}} \Phi^o(0, D_t\chi_E) &\leq \int_{\widehat{A}\times\mathbb{R}} \Phi^o(0, D_t\chi_F). \end{aligned} \tag{2.27}$$

Then for any $\widehat{A} \in \operatorname{Op}_b(\widehat{\Omega})$ we have

$$\int_{\widehat{A}\times\mathbb{R}} \Phi^o(D_x\chi_E, D_t\chi_E) \leq \int_{\widehat{A}\times\mathbb{R}} \Phi^o(D_x\chi_F, D_t\chi_F). \tag{2.28}$$

Proof. We may assume that $\int_{\widehat{A}\times\mathbb{R}} \Phi^o(D_x\chi_F, D_t\chi_F) < +\infty$. It then follows that $\int_{\widehat{A}\times\mathbb{R}} \Phi^o(D_x\chi_E, D_t\chi_E) < +\infty$. Indeed, since all norms in $\mathbb{R}^{n+1}$ are comparable, there exists $c, C > 0$ such that

$$c\Phi^o(\xi) \leq \Phi^o(\widehat{\xi}, 0) + \Phi^o(0, \xi_{n+1}) \leq C\Phi^o(\xi), \quad \xi \in \mathbb{R}^{n+1},$$

thus

$$\begin{aligned} c\int_{\widehat{A}\times\mathbb{R}} \Phi^o(D_x\chi_E, D_t\chi_E) &\leq \int_{\widehat{A}\times\mathbb{R}} \Phi^o(D_x\chi_E, 0) + \int_{\widehat{A}\times\mathbb{R}} \Phi^o(0, D_t\chi_E) \\ \leq \int_{\widehat{A}\times\mathbb{R}} \Phi^o(D_x\chi_F, 0) &+ \int_{\widehat{A}\times\mathbb{R}} \Phi^o(0, D_t\chi_F) \leq C\int_{\widehat{A}\times\mathbb{R}} \Phi^o(D_x\chi_F, D_t\chi_F). \end{aligned}$$

By definition of Φ-perimeter and Proposition 2.29, for any $\varepsilon > 0$ there exists $\eta \in C_c(\widehat{A} \times \mathbb{R}; B_\Phi)$ such that $\Phi(\eta) = \omega(\varphi(\widehat{\eta}), |\eta_{n+1}|) \leq 1$ and

$$\int_{\widehat{A}\times\mathbb{R}} \Phi^o(D\chi_E) - \varepsilon < -\int_{\widehat{A}\times\mathbb{R}} \operatorname{div} \eta dx = \int_{\widehat{A}\times\mathbb{R}} \eta \cdot D\chi_E. \tag{2.29}$$

Then from (2.27), (2.22), (2.24) and (2.23) we get

$$\begin{aligned}
\int_{\widehat{A}\times\mathbb{R}} \eta\cdot D\chi_E &= \int_{\widehat{A}\times\mathbb{R}} \left(\sum_{j=1}^{n} \eta_j \cdot D_{x_j}\chi_E + \eta_{n+1} D_t\chi_E\right) \\
&\le \int_{\widehat{A}\times\mathbb{R}} \left(\varphi(\widehat{\eta})\, d\varphi^o(D_x\chi_E) + |\eta_{n+1}|\, d|D_t\chi_E|\right) \\
&\le \int_{\widehat{A}\times\mathbb{R}} \left(\varphi(\widehat{\eta})\, d\varphi^o(D_x\chi_F) + |\eta_{n+1}|\, d|D_t\chi_F|\right) \\
&\le \int_{\widehat{A}\times\mathbb{R}} \omega(\varphi(\widehat{\eta}), |\eta_{n+1}|)\, d\omega^o(\varphi^o(D_x\chi_F), |D_t\chi_F|) \\
&\le \int_{\widehat{A}\times\mathbb{R}} \omega^o(\varphi^o(D_x\chi_F), |D_t\chi_F|)) = \int_{\widehat{A}\times\mathbb{R}} \Phi^o(D\chi_F).
\end{aligned}$$

This inequality, (2.29) and the arbitrariness of ε yield (2.28).

□

Theorem 2.32 (Minimizers and cartesian minimizers). *Let* $\Phi : \mathbb{R}^{n+1} \to [0, +\infty)$ *be a norm, and* $u \in BV_{\mathrm{loc}}(\widehat{\Omega})$. *The following assertions hold:*

(a) if $\mathrm{sg}(u)$ *is a minimizer of* P_Φ *in* $\widehat{\Omega}\times\mathbb{R}$, *then* u *is a minimizer of* $\mathcal{G}_{\Phi^o}$ *in* $\widehat{\Omega}$*;*

(b) if Φ *is partially monotone and* u *is a minimizer of* $\mathcal{G}_{\Phi^o}$ *in* $\widehat{\Omega}$, *then* $\mathrm{sg}(u)$ *is a minimizer of* P_Φ *in* $\Omega = \widehat{\Omega}\times\mathbb{R}$.

Proof. (a) Let $\psi \in C^1_c(\widehat{\Omega})$ be such that $\mathrm{supp}\,(\psi) \subset\subset \widehat{A}$ for some $\widehat{A} \in \mathrm{Op}_b(\widehat{\Omega})$. Then there exists $H > 0$ such that $\mathrm{sg}(u)\Delta\mathrm{sg}(u+\psi) \subset\subset \widehat{A}\times(-H, H)$. If $\mathrm{sg}(u)$ is a minimizer of P_Φ, then $P_\Phi(\mathrm{sg}(u), \widehat{A}\times\mathbb{R}) \le P_\Phi(\mathrm{sg}(u+\psi), \widehat{A}\times\mathbb{R})$ and so, by virtue of (2.19),

$$\mathcal{G}_{\Phi^o}(u, \widehat{A}) \le \mathcal{G}_{\Phi^o}(u+\psi, \widehat{A}). \tag{2.30}$$

For general $\psi \in BV_{\mathrm{loc}}(\widehat{\Omega})$ inequality (2.30) can be proven by approximation.

(b) Let u be a minimizer of $\mathcal{G}_{\Phi^o}$ and $F \in BV_{\mathrm{loc}}(\Omega)$ be such that $\mathrm{sg}(u)\Delta F \subset\subset A = \widehat{A}\times(-M, M)$ with $\widehat{A} \in \mathrm{Op}_b(\widehat{\Omega})$ and $M > 0$. Then (2.2) yields that

$$P_\Phi(F, \widehat{B}\times\mathbb{R}) < +\infty \qquad \forall \widehat{B} \in \mathrm{Op}_b(\widehat{\Omega}).$$

We shall closely follow [63, 88], where the argument is done in the Euclidean setting. For simplicity let $\varphi^o_1(\cdot) = \Phi^o(\cdot, 0)$ and $\varphi^o_2(\cdot) = \Phi^o(0, \cdot)$. We claim that there exists $v \in BV_{\mathrm{loc}}(\widehat{\Omega})$ with $\mathrm{supp}\,(u - v) \subset\subset \widehat{A}$ such that for any $\widehat{B} \in \mathrm{Op}_b(\widehat{\Omega})$

$$\begin{aligned}
\int_{\widehat{B}\times\mathbb{R}} \varphi^o_1(D_x\chi_{\mathrm{sg}(v)}) &\le \int_{\widehat{B}\times\mathbb{R}} \varphi^o_1(D_x\chi_F), \\
\int_{\widehat{B}\times\mathbb{R}} \varphi^o_2(D_t\chi_{\mathrm{sg}(v)})) &\le \int_{\widehat{B}\times\mathbb{R}} \varphi^o_2(D_t\chi_F).
\end{aligned} \tag{2.31}$$

Supposing that the claim is true, from (2.31) and from Lemma 2.31 we deduce

$$P_\Phi(\mathrm{sg}(v), \widehat{A}\times\mathbb{R}) = \int_{\widehat{A}\times\mathbb{R}} \Phi^o(D\chi_{\mathrm{sg}(v)}) \le \int_{\widehat{A}\times\mathbb{R}} \Phi^o(D\chi_F) = P_\Phi(F, \widehat{A}\times\mathbb{R}).$$

Then by the minimality of u and (2.19) we get

$$\begin{aligned} P_\Phi(\mathrm{sg}(u), \widehat{A}\times\mathbb{R}) = \int_{\widehat{A}} \Phi^o(-Du,1) \le \int_{\widehat{A}} \Phi^o(-Dv,1) \\ = P_\Phi(\mathrm{sg}(v), \widehat{A}\times\mathbb{R}) \le P_\Phi(F, \widehat{A}\times\mathbb{R}). \end{aligned}$$

Let us prove our claim. Since $\mathrm{sg}(u)\Delta F \subset\subset A,$ we have

$$\lim_{t\to+\infty}\chi_F(x,t)=0, \quad \lim_{t\to-\infty}\chi_F(x,t)=1 \quad \text{for a.e. } x\in\widehat{\Omega}. \tag{2.32}$$

Then, by [63, Lemma 14.7 and Theorem 14.8] (see also [88, Theorem 2.3]) the function

$$v_h(x) := \int_{-h}^{h}\chi_F(x,t)dt - h, \qquad x\in\widehat{\Omega},$$

belongs to $L^1_{\mathrm{loc}}(\widehat{\Omega})$ and the sequence $\{v_h\}$ converges pointwise to $v\in BV_{\mathrm{loc}}(\widehat{\Omega})$ as $h\to+\infty.$ To show that $u-v$ is compactly supported in $\widehat{A}$ it is enough to take $\widehat{A}'\in \mathrm{Op}_b(\widehat{\Omega})$ such that $\widehat{A}'\subset\subset\widehat{A}$ and $\mathrm{sg}(u)\Delta F\subset\subset \widehat{A}'\times(-M,M),$ and to observe that since $\mathrm{sg}(u)\cap\big((\widehat{\Omega}\setminus\widehat{A}')\times\mathbb{R}\big) = F\cap\big((\widehat{\Omega}\setminus\widehat{A}')\times\mathbb{R}\big),$ if $x\in\widehat{\Omega}\setminus\widehat{A}',$ for h sufficiently large we have

$$v_h(x) = \int_{-h}^{u(x)}\chi_F(x,t)dt - h = u(x).$$

Now, define $\eta_h:\mathbb{R}\to[0,+\infty)$ as $\eta_h:=1$ on $[-h,h],$ $\eta_h:=0$ on $(-\infty,-h-1]\cup[h+1,+\infty),$ and

$$\eta_h(t) := \begin{cases} h+1-t & \text{if } h\le t\le h+1,\\ h+1+t & \text{if } -h-1\le t\le -h.\end{cases}$$

Being $1/2=\int_{-h-1}^{-h}[h+1+t]dt,$ we have

$$\begin{aligned} \Big|\int_{\mathbb{R}}\eta_h(t)\chi_F(x,t)dt - h - \frac12 - v(x)\Big| &= \Big| -\int_{-h-1}^{-h}[h+1+t](1-\chi_F(x,t))dt \\ &\quad + \int_h^{h+1}[h+1-t]\chi_F(x,t)dt + v_h(x)-v(x)\Big| \\ &\le \Big|v_h(x)-v(x)\Big| + \int_{-h-1}^{-h}(1-\chi_F(x,t))dt + \int_h^{h+1}\chi_F(x,t)dt. \end{aligned}$$

Hence, from (2.32), for almost every $x\in\widehat{\Omega}$ we get

$$\lim_{h\to+\infty}\Big|\int_{\mathbb{R}}\eta_h(t)\chi_F(x,t)dt - h - \frac12 - v(x)\Big| = 0. \tag{2.33}$$

Let us fix $\psi \in C_c^1(\widehat{\Omega})$ and $1 \le j \le n$. Then, using $\int_{\widehat{\Omega}} D_{x_j}\psi(x)dx = 0$, the dominated convergence theorem (see [63, 88] for more details) and (2.33) we find

$$\begin{aligned}\int_{\widehat{\Omega}\times\mathbb{R}} \psi(x) D_{x_j}\chi_F(x,t) &= \lim_{h\to+\infty} \int_{\widehat{\Omega}\times\mathbb{R}} \eta_h(t)\psi(x) D_{x_j}\chi_F(x,t) \\ &= -\lim_{h\to+\infty} \int_{\widehat{\Omega}} D_{x_j}\psi(x)dx \int_{\mathbb{R}} \eta_h(t)\chi_F(x,t)dt \\ &= -\lim_{h\to+\infty} \int_{\widehat{\Omega}} D_{x_j}\psi(x) \left[\int_{\mathbb{R}} \eta_h(t)\chi_F(x,t)dt - h - \frac{1}{2}\right] dx \\ &= -\int_{\widehat{\Omega}} v(x) D_{x_j}\psi(x)dx.\end{aligned}$$

Hence for any $\widehat{A} \in \mathrm{Op}_b(\widehat{\Omega})$ and $\eta \in C_c^1(\widehat{A}; B_{\varphi_1})$ one has

$$-\int_{\widehat{A}} v(x) \sum_{j=1}^{n} D_{x_j}\eta(x)dx = \int_{\widehat{A}\times\mathbb{R}} \eta(x)\cdot D_x\chi_F(x,t) \le \int_{\widehat{A}\times\mathbb{R}} \varphi_1^o(D_x\chi_F(x,t)).$$

Since η is arbitrary, the definition of $\int_{\widehat{A}} \varphi_1^o(D_x v)$ implies

$$\int_{\widehat{A}} \varphi_1^o(D_x v) \le \int_{\widehat{A}\times\mathbb{R}} \varphi_1^o(D_x\chi_F). \tag{2.34}$$

Being $|D_t\chi_F|$ a counting measure, we have [88]

$$\int_{\widehat{A}\times\mathbb{R}} \varphi_2^o(D_t\chi_F) = \varphi_2^o(1) \int_{\widehat{A}\times\mathbb{R}} |D_t\chi_F| \ge \varphi_2^o(1)|\widehat{A}|.$$

Moreover, one checks that

$$\int_{\widehat{A}\times\mathbb{R}} \varphi_1^o(D_x\chi_{\mathrm{sg}(v)}) = \int_{\widehat{A}} \varphi_1^o(D_x v), \qquad \int_{\widehat{A}\times\mathbb{R}} \varphi_2^o(D_t\chi_{\mathrm{sg}(v)}) = \varphi_2^o(1)\,|\widehat{A}|. \tag{2.35}$$

Now our claim (2.31) follows from (2.34)-(2.35). □

Corollary 2.33. *Let $\Phi^o : \mathbb{R}^{n+1} \to [0,+\infty)$ be a partially monotone norm and $u \in \mathcal{M}_{\Phi^o}(\widehat{\Omega})$. Then $u \in L^\infty_{\mathrm{loc}}(\widehat{\Omega})$.*

Proof. It follows repeating essentially the same arguments in the proof of [63, Theorem 14.10], using Theorem 2.32(b) and the density estimates (see for instance [85, Proposition 1.10] for the anisotropic setting). □

2.4 Classification of cartesian minimizers for cylindrical norms

The aim of this section is to give a rather complete classification of entire cartesian minimizers, supposing the norm Φ cylindrical. As explained in the introduction, this case covers, in particular, the study of minimizers of the total variation functional. We start with a couple of observations.

Remark 2.34. Suppose that $\Phi : \mathbb{R}^{n+1} \to [0, +\infty)$ is cylindrical over φ. Then

$$u \in \mathcal{M}_{\Phi^o}(\widehat{\Omega}) \Longrightarrow \lambda u \in \mathcal{M}_{\Phi^o}(\widehat{\Omega}) \qquad \forall \lambda \in \mathbb{R}, \tag{2.36}$$

since

$$\mathcal{G}_{\Phi^o}(v, \widehat{A}) = \int_{\widehat{A}} \varphi^o(Dv) + |\widehat{A}|, \qquad (v, \widehat{A}) \in BV_{\mathrm{loc}}(\widehat{\Omega}) \times \mathrm{Op}_b(\widehat{\Omega}).$$

On the other hand, (2.36) is expected to hold not for all non cylindrical norms Φ. For example, let Φ be Euclidean, $n \geq 8$ and $u : \mathbb{R}^n \to \mathbb{R}$ be a smooth nonlinear solution [27] of the minimal surface equation $\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0$. Then[6] $u \in \mathcal{M}_{\Phi^o}(\mathbb{R}^n)$, but if $\Delta u |\nabla u|^2$ is not identically zero, then $\lambda u \notin \mathcal{M}_{\Phi^o}(\mathbb{R}^n)$ for any $\lambda \in \mathbb{R} \setminus \{0, \pm 1\}$. Indeed, otherwise λu solves the minimal surface equation, hence

$$\begin{aligned}
0 =& \lambda \mathrm{div}\left(\frac{\nabla u}{\sqrt{1+\lambda^2|\nabla u|^2}}\right) \\
=& \frac{\lambda}{\sqrt{1+\lambda^2|\nabla u|^2}}\left(\Delta u - \frac{\lambda^2}{1+\lambda^2|\nabla u|^2}\sum_{i,j=1}^{n} \nabla_i u \cdot \nabla_j u \nabla_{ij} u\right) \\
=& \frac{\lambda}{\sqrt{1+\lambda^2|\nabla u|^2}}\left(\Delta u - \lambda^2 \Delta u \frac{1+|\nabla u|^2}{1+\lambda^2|\nabla u|^2}\right) = \frac{\lambda(1-\lambda^2)\Delta u\, |\nabla u|^2}{(1+\lambda^2|\nabla u|^2)^{3/2}}.
\end{aligned}$$

If $\Delta u |\nabla u|^2$ is not identically zero, we get $\lambda(1-\lambda^2) = 0$, a contradiction.

Remark 2.35. Suppose that $\Phi : \mathbb{R}^{n+1} \to [0, +\infty)$ is cylindrical over φ. Then

$$u \in \mathcal{M}_{\Phi^o}(\widehat{\Omega}) \implies \max\{u, \lambda\}, \min\{u, \lambda\} \in \mathcal{M}_{\Phi^o}(\widehat{\Omega}) \quad \forall \lambda \in \mathbb{R}. \tag{2.37}$$

Indeed, suppose first $\lambda = 0$. If $\{u \geq 0\} \in BV_{\mathrm{loc}}(\widehat{\Omega})$, then (2.37) can be proven as in [93, Lemma 3.5], using [11, Theorem 3.84]. In the general case, by the coarea formula there exists a sequence $\lambda_j \uparrow 0$ such that $\{u \geq \lambda_j\} \in BV_{\mathrm{loc}}(\widehat{\Omega})$. Clearly $u_j := u - \lambda_j \in \mathcal{M}_{\Phi^o}(\widehat{\Omega})$, hence $u_j^+ \in \mathcal{M}_{\Phi^o}(\widehat{\Omega})$. Since $u_j^+ \to u^+$ in $L^1_{\mathrm{loc}}(\widehat{\Omega})$, Theorem 2.26 implies $u^+ \in \mathcal{M}_{\Phi^o}(\widehat{\Omega})$. The case $\lambda \neq 0$ is implied by the previous proof and the identity $\max\{u, \lambda\} = (u-\lambda)^+ + \lambda$. The relation $\min\{u, \lambda\} \in \mathcal{M}_{\Phi^o}(\widehat{\Omega})$ then follows from the identity $\min\{u, \lambda\} = -\max\{-u, -\lambda\}$ and from Remark 2.34.

Further properties of cartesian minimizers are listed in the following proposition, which in particular (when φ is Euclidean) asserts some properties of minimizers of the total variation functional [38].

Proposition 2.36 (Cartesian minimizers for cylindrical norms). *Suppose that $\Phi : \mathbb{R}^{n+1} \to [0, +\infty)$ is cylindrical over φ. The following assertions hold:*

(a) if $u \in \mathcal{M}_{\Phi^o}(\widehat{\Omega})$ and $\lambda \in \mathbb{R}$ then $\chi_{\{u>\lambda\}}, \chi_{\{u\geq\lambda\}} \in \mathcal{M}_{\Phi^o}(\widehat{\Omega})$;

[6] u is a minimizer of $\mathcal{G}_{\Phi^o}$ in $\mathbb{R}^n$, since the Euclidean unit normal (pointing upwards) to $\mathrm{graph}(u)$, constantly extended in the e_{n+1} direction, provides a calibration for $\mathrm{graph}(u)$ in the whole of $\mathbb{R}^{n+1}$.

(b) *if* $\widehat{E} \subset \widehat{\Omega}$ *and* $\chi_{\widehat{E}} \in \mathcal{M}_{\Phi^o}(\widehat{\Omega})$ *then* $\widehat{E}$ *is a minimizer of* P_φ *in* $\widehat{\Omega}$;

(c) *if* $u \in \mathcal{M}_{\Phi^o}(\widehat{\Omega})$ *and* $\lambda \in \mathbb{R}$ *then* $\{u > \lambda\}$ *and* $\{u \geq \lambda\}$ *are minimizers of* P_φ *in* $\widehat{\Omega}$*;*

(d) *if* $u \in BV_{\mathrm{loc}}(\widehat{\Omega})$ *and for almost every* $\lambda \in \mathbb{R}$ *the sets* $\{u > \lambda\}$ *(resp.* $\{u \geq \lambda\}$*) are minimizers of* P_φ *in* $\widehat{\Omega}$*, then* $u \in \mathcal{M}_{\Phi^o}(\widehat{\Omega})$*;*

(e) *if* $u \in \mathcal{M}_{\Phi^o}(\widehat{\Omega})$ *and* $f : \mathbb{R} \to \mathbb{R}$ *is monotone then* $f \circ u \in \mathcal{M}_{\Phi^o}(\widehat{\Omega})$*;*

(f) *let* $\zeta \in \mathbb{R}^n$*,* $f : \mathbb{R} \to \mathbb{R}$ *be a monotone function, and define* $u(x) := f(x \cdot \zeta)$ *for any* $x \in \widehat{\Omega}$*. Then* $u \in \mathcal{M}_{\Phi^o}(\widehat{\Omega})$.

Clearly, assertion (e) generalizes (2.36) and (2.37). We also anticipate here that the converse of statement (f) is considered in Theorem 2.41 below.

Proof. The proof of (a) is the same as in [27, Theorem 1] and (b) is immediate. (c) follows from (a) and (b), while (d) follows from the coarea formula

$$\int_{\widehat{A}} \varphi^o(Dv) = \int_{\mathbb{R}} P_\varphi(\{v > \lambda\}, \widehat{A})\, d\lambda, \qquad v \in BV(\widehat{A}).$$

Let us prove (e). Without loss of generality we can assume that f is nondecreasing. For each $\lambda \in \mathbb{R}$ define

$$c(\lambda) := \inf\{t \in \mathbb{R} : \ f(t) > \lambda\}.$$

Letting $v := f \circ u$, one can check that

$$\{v > \lambda\} = \begin{cases} \{u > c(\lambda)\} & \text{if } f(c(\lambda)) \leq \lambda, \\ \{u \geq c(\lambda)\} & \text{if } f(c(\lambda)) > \lambda. \end{cases}$$

Since $u \in \mathcal{M}_{\Phi^o}(\widehat{\Omega})$, by (c) both $\{u > c(\lambda)\}$ and $\{u \geq c(\lambda)\}$ are minimizers of P_φ in $\widehat{\Omega}$, i.e. $\{v > \lambda\}$ is a minimizer of P_φ in $\widehat{\Omega}$ for all $\lambda \in \mathbb{R}$. Then (d) implies that $v \in \mathcal{M}_{\Phi^o}(\widehat{\Omega})$.

Finally (f) follows from (e), since the linear function $u_0(x) = x \cdot \zeta$, $x \in \widehat{\Omega}$, is a minimizer of $\mathcal{G}_{\Phi^o}$ in $\widehat{\Omega}$. □

Now, we show that Proposition 2.36(f) implies the minimality of certain cones; the same conclusion could be obtained by applying Remark 2.12.

Proposition 2.37 (**Cones minimizing the anisotropic perimeter**)**.** *Suppose that* $\Phi : \mathbb{R}^{n+1} \to [0, +\infty)$ *is cylindrical over* φ*. Let* $H_1, H_2 \subset \mathbb{R}^{n+1}$ *be two half-spaces, with outer unit normals* $\nu_1, \nu_2 \in \mathbb{S}^n$ *respectively. Suppose that*

$$\{0\} \in \partial H_1 \cap \partial H_2 \subset \{t = 0\}, \tag{2.38}$$

and that

(a) $\nu_1 \cdot \nu_2 \geq 0$, $\nu_2 \cdot e_{n+1} \geq \nu_1 \cdot e_{n+1} \geq 0$*;*

(b) $\arccos(\nu_1 \cdot \nu_2) + \arccos(\nu_2 \cdot e_{n+1}) = \arccos(\nu_1 \cdot e_{n+1})$.

Then the cones $E := H_1 \cap H_2$ *and* $F := H_1 \cup H_2$ *are minimizers of* P_Φ *in* $\mathbb{R}^{n+1}$.

Before proving the proposition, some comments are in order. Our assumptions on H_1 and H_2 exclude, in particular, that E is a "roof-like" cone (as the one depicted in Figure 2.4). More specifically, in case $\nu_1 \neq \nu_2$, the inclusion $\partial H_1 \cap \partial H_2 \subset \{t = 0\}$ in (2.38) implies that the orthogonal complement to $\{t = 0\}$ is contained in the span of the orthogonal complements of ∂H_i, *i.e.*

$$e_{n+1} \in \operatorname{span}(\nu_1, \nu_2).$$

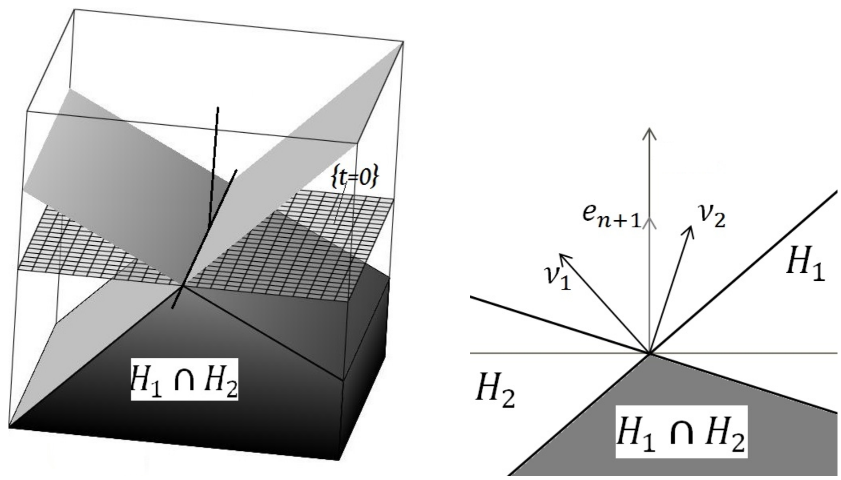

Figure 2.4: "Roof" like cone (left) and its section (right) along $(\partial H_1 \cap \partial H_2)^\perp$.

Next, assumption (a) implies that ν_1 and ν_2 lie "on the same side" with respect to e_{n+1}, while assumption (b) implies that ν_2 lies between ν_1 and e_{n+1} (a condition not satisfied in Figure 2.4, and satisfied in Figure 2.5). We shall see in Example 2.39 that, if condition (b) is not satisfied, then E and F need not be minimizers.

Proof. For $i = 1, 2$, define

$$\lambda_i := \begin{cases} \dfrac{\sqrt{1 - (\nu_i \cdot e_{n+1})^2}}{\nu_i \cdot e_{n+1}} = \tan \alpha_i & \text{if } \nu_i \cdot e_{n+1} \neq 0, \\ +\infty & \text{if } \nu_i \cdot e_{n+1} = 0, \end{cases}$$

see Figure 2.5, left. By (a) we have $\lambda_2 \leq \lambda_1$. Let $\widehat{\nu} \in \mathbb{S}^{n-1} \subset \{t = 0\}$ be a unit normal to $\partial H_1 \cap \partial H_2$ which, according to (b), can be chosen so that $(\widehat{\nu}, 0) \cdot \nu_i \leq 0, \quad i = 1, 2$.

If $\lambda_2 = +\infty$, then by conditions (a) and (b) we have $H_1 = H_2 = H$, where H is the half-space whose outer unit normal is $-(\widehat{\nu}, 0)$. By Example 2.7 it follows that $H = E = F$ is a minimizer of P_Φ.

Assume that $\lambda_2 \leq \lambda_1 < +\infty$. Define

$$f(\sigma) := \begin{cases} \lambda_2 \sigma, & \sigma \geq 0 \\ \lambda_1 \sigma, & \sigma < 0 \end{cases}, \qquad g(\sigma) := \begin{cases} \lambda_1 \sigma, & \sigma \geq 0 \\ \lambda_2 \sigma, & \sigma < 0 \end{cases}.$$

Then $E = \mathrm{sg}(u)$, $F = \mathrm{sg}(v)$, where $u(x) := f(x \cdot \widehat{\nu})$, $v(x) := g(x \cdot \widehat{\nu})$, $x \in \mathbb{R}^n$. Since f, g are *monotone*, by Proposition 2.36(f) we have $u, v \in \mathcal{M}_{\Phi^o}(\mathbb{R}^n)$. Since Φ^o is partially monotone (recall Example 2.28), Theorem 2.32(b) yields that E and F are minimizers of P_Φ in $\mathbb{R}^{n+1}$. Now, assume that $0 \le \lambda_2 < \lambda_1 = +\infty$. Then $\nu_1 = -(\widehat{\nu}, 0)$. We prove that F is a minimizer of

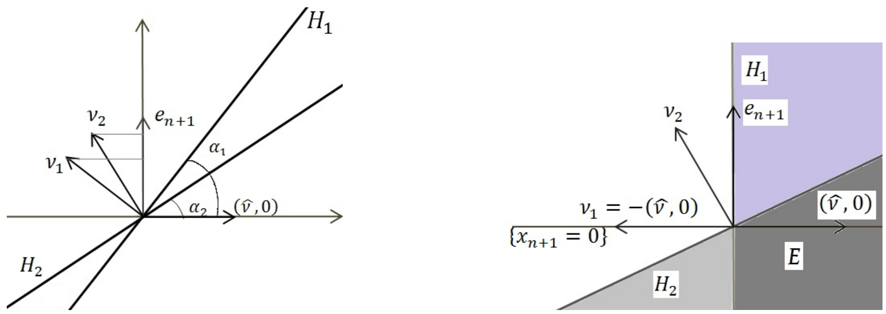

Figure 2.5: Sections of cones when $\lambda_1 < +\infty$ and $\lambda_1 = +\infty$.

P_Φ in $\mathbb{R}^{n+1}$ (the proof for E being similar). It is enough to show minimality of F inside every strip $S_m = \mathbb{R}^n \times (-m, m)$, $m > 0$. Define $h : \mathbb{R} \to \mathbb{R}$ as $h(\sigma) := m\chi_{(0,+\infty)}(\sigma)$ if $\lambda_2 = 0$ and

$$h(\sigma) := \begin{cases} -m & \text{if } \sigma < -\frac{m}{\lambda_2}, \\ \lambda_2 \sigma & \text{if } -\frac{m}{\lambda_2} \le \sigma < 0, \\ m & \text{if } \sigma \ge 0 \end{cases}$$

if $\lambda_2 > 0$. Let $w(x) := h(x \cdot \widehat{\nu})$, $x \in \mathbb{R}^n$. As before, the subgraph $\mathrm{sg}(w)$ of w is a minimizer of P_Φ in $\mathbb{R}^{n+1}$. Since $\mathrm{sg}(w) \cap S_m = F \cap S_m$, it follows that F is a minimizer of P_Φ in S_m. $\square$

Remark 2.38. It is not difficult to see that in Proposition 2.37 the assumption $\partial H_1 \cap \partial H_2 \subset \{t = 0\}$ is in general not necessary. Indeed, assume $n = 2$, $\Phi^o(\xi_1^*, \xi_2^*, \xi_3^*) = |\xi_1^*| + |\xi_2^*| + |\xi_3^*|$ and H_i, $i = 1, 2$ are half-spaces with outer unit normals $\nu_1 = (\frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{1}{2})$, $\nu_2 = (\frac{1}{\sqrt{10}}, \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{10}})$ respectively. Then both $H_1 \cap H_2$ and $H_1 \cup H_2$ are minimizers of P_Φ in $\mathbb{R}^3$. Indeed, for the Euclidean isometry $U(x, t) := (x_1, t, x_2)$, one sees that UH_1 and UH_2 satisfy the assumptions of Proposition 2.37, hence $UH_1 \cap UH_2$ and $UH_1 \cup UH_2$ are minimizers of P_Φ. Since $\Phi \circ U = \Phi$, the thesis follows.

Example 2.39 (Non minimal cones). Let $n = 1$, $\Phi^o(\xi_1^*, \xi_2^*) = |\xi_1^*| + |\xi_2^*|$ and H_1 and H_2 be half-planes of $\mathbb{R}^2$ with outer unit normals $\nu_1, \nu_2 \in \mathbb{S}^1$ such that

(a) $\partial H_1 \cap \partial H_2 = \{0\}$;

(b) $\nu_2 \neq e_2$, $\nu_1 \cdot \nu_2 \ge 0$, and $\nu_2 \cdot e_2 \ge \nu_1 \cdot e_2 \ge 0$;

(c) $\arccos(\nu_1 \cdot \nu_2) = \arccos(\nu_1 \cdot e_2) + \arccos(e_2 \cdot \nu_2)$.

Then the cones $E := H_1 \cap H_2$ and $F := H_1 \cup H_2$ are not minimizers of P_Φ. Let us prove the assertion for E, the statement for F being similar. The lines ∂H_1, ∂H_2 and $\{t = -1\}$ compose a nondegenerate triangle $T \subset E$ with sides $a_1, a_2, b > 0$, b the horizontal side. For any $A \in \mathrm{Op}_b(\mathbb{R}^2)$ with $T \subset\subset A$ we have

$$P_\Phi(E, A) - P_\Phi(E \setminus T, A) = a_1 \Phi^o(\nu_1) + a_2 \Phi^o(\nu_2) - b\Phi^o(e_2) \ge a_1 + a_2 - b > 0,$$

since $\Phi^o(\nu) \ge 1$ for all $\nu \in \mathbb{S}^1$. Hence, E is not a minimizer of P_Φ.

We shall need the following relevant result (see for instance [63, Theorem 17.3] and references therein).

Theorem 2.40. *Let $\widehat{E}$ be a minimizer of the Euclidean perimeter in $\mathbb{R}^n$. Then either $n \geq 8$ or $\partial\widehat{E}$ is a hyperplane.*

Our classification result of minimizers of $\mathcal{G}_{\Phi^o}$ reads as follows:

Theorem 2.41 (Entire cartesian minimizers). *Suppose that $\Phi : \mathbb{R}^{n+1} \to [0, +\infty)$ is cylindrical over φ. Assume one of the two following alternatives:*

(a) $1 \leq n \leq 7$ and φ is Euclidean;

(b) $n = 2$ and φ^o is strictly convex.

If u is a minimizer of $\mathcal{G}_{\Phi^o}$ in $\mathbb{R}^n$ then there exists $\zeta \in \mathbb{S}^{n-1}$ and a monotone function $f : \mathbb{R} \to \mathbb{R}$ such that

$$u(x) = f(x \cdot \zeta), \qquad x \in \mathbb{R}^n. \tag{2.39}$$

Remark 2.42. If φ is a noneuclidean smooth and uniformly convex norm, the conclusion of Theorem 2.41 under assumption (a) does not necessarily hold. For example, if $n = 4$ and K is the cone over the Clifford torus [90] – a minimizer of P_φ in $\mathbb{R}^4$ for some uniformly convex smooth norm φ – then by Proposition 2.36(d), $u = \chi_K$ is a minimizer of $\mathcal{G}_{\Phi^o}$ in $\mathbb{R}^4$ which cannot be represented as in (2.39). We don't know if there are counterexamples also for $n = 3$.

Proof. Let $u \in \mathcal{M}_{\Phi^o}(\mathbb{R}^n)$. By Corollary 2.33, $u \in L^\infty_{\rm loc}(\mathbb{R}^n)$. Let

$$c_0 := \operatorname*{ess\,inf}_{x\in\mathbb{R}^n} u(x) \in [-\infty, +\infty), \qquad c_1 := \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} u(x) \in (-\infty, +\infty].$$

If $c_0 = c_1$, then $u \equiv c_0$ a.e. on $\mathbb{R}^n$. In this case $\zeta \in \mathbb{S}^{n-1}$ can be chosen arbitrarily and $f \equiv c_0$.

Assume that $-\infty \leq c_0 < c_1 \leq +\infty$. Given $\lambda \in \mathbb{R}$, Proposition 2.36(c) implies that $\{u > \lambda\}$ is a minimizer of P_φ in $\mathbb{R}^n$. We claim that either $\partial^*\{u > \lambda\}$ is a hyperplane or $\partial^*\{u > \lambda\} = \emptyset$. Indeed, if $n = 1$ the claim is trivial. If $n = 2$ and φ strictly convex, the claim follows from [94, Theorem 3.11]. When $3 \leq n \leq 7$ and φ is Euclidean, the claim is implied by Theorem 2.40. Hence for any $\lambda \in (c_0, c_1)$ there exist $\zeta_\lambda \in \mathbb{S}^{n-1}$ and $a_\lambda \in \mathbb{R}$ such that

$$\{u > \lambda\} = \{x \in \mathbb{R}^n : \ x \cdot \zeta_\lambda < a_\lambda\}.$$

In addition, these hyperplanes cannot intersect transversely, hence there exists $\zeta \in \mathbb{S}^{n-1}$ such that $\zeta_\lambda = \zeta$ for all $\lambda \in (c_0, c_1)$. Since the function $\lambda \in (c_0, c_1) \mapsto a_\lambda$ is monotone, it remains to construct the function f. We may assume that $\lambda \mapsto a_\lambda$ is nonincreasing, the nondecreasing case being similar. Extend a_λ to $\mathbb{R} \setminus [c_0, c_1]$ setting $a_\lambda := +\infty$ for $\lambda < c_0$ if $c_0 \in \mathbb{R}$, and $a_\lambda := -\infty$ for $\lambda > c_1$ if $c_1 \in \mathbb{R}$. Then, we define

$$f(\sigma) := \sup\{\lambda : \ \sigma < a_\lambda\}, \quad \sigma \in \mathbb{R},$$

which is nonincreasing. Note that f is real valued. Indeed, if $f(\sigma) = -\infty$ for some $\sigma \in \mathbb{R}$, then $\sigma \geq a_\lambda$ for all $\lambda \in \mathbb{R}$ which is impossible since $a_\lambda \to +\infty$ as $\lambda \to -\infty$. Similarly, $f(\sigma) < +\infty$ for any $\sigma \in \mathbb{R}$.

Set $v(x) := f(x \cdot \zeta)$. By construction, we have $\{v > \lambda\} = \{u > \lambda\}$ for a.e. $\lambda \in \mathbb{R}$. It is easy to check that if $w \in L^1_{\rm loc}(\mathbb{R}^n)$ then for a.e. $x \in \mathbb{R}^n$ one has

$$w(x) = \int_0^{+\infty} \chi_{\{w>\lambda\}}(x) d\lambda + \int_{-\infty}^0 (1 - \chi_{\{w>\lambda\}}(x)) d\lambda,$$

hence $u = v$ almost everywhere on $\mathbb{R}^n$. □

Remark 2.43. It seems not easy to generalize Theorem 2.41 to noneuclidean φ (for some $n \in \{3, \ldots, 7\}$)[7], since our argument was based on Theorem 2.40.

Remark 2.44. Assumption (a) of Theorem 2.41 is optimal in the sense that if $n \geq 8$ there exist minimizers of $\mathcal{G}_{\Phi^o}$ on $\mathbb{R}^n$ which cannot be written as in (2.39). Indeed, let $C \subset \mathbb{R}^8$ be the Simons cone minimizing the Euclidean perimeter [27, Theorem A]. By Proposition 2.36(d) $u = \chi_C \in \mathcal{M}_{\Phi^o}(\mathbb{R}^n)$, however u does not admit the representation (2.39).

From Theorem 2.41 and Proposition 2.36(f) we deduce the following result.

Corollary 2.45 (Composition of linear and monotone functions). *Under the assumptions of Theorem 2.41, u is a minimizer of $\mathcal{G}_{\Phi^o}$ in $\mathbb{R}^n$ if and only if there exists $\zeta \in \mathbb{S}^{n-1}$ and a monotone function $f : \mathbb{R} \to \mathbb{R}$ such that $u(x) = f(x \cdot \zeta)$ for any $x \in \mathbb{R}^n$.*

2.5 Lipschitz regularity of cartesian minimizers for cylindrical norms

We recall from [93, Theorem 3.12] that if $n = 2$ and if ∂B_φ either does not contain segments, or it is locally a graph in a neighborhood of its segments, then the graph of a minimizer of $\mathcal{G}_{\Phi^o}$ in $\mathbb{R}^2$ is locally Lipschitz. On the other hand, an example in [93, Sect. 4] shows that such a regularity result cannot be expected for a general anisotropy. More precisely, for Φ^o cylindrical as in (2.6) with $\varphi^o(\widehat{\xi^*}) = |\xi_1^*| + |\xi_2^*|$, that example exhibits a function $u \in \mathcal{M}_{\Phi^o}(\mathbb{R}^2)$ such that the set of points where the boundary of $\mathrm{sg}(u)$ is not locally the graph of a Lipschitz function has positive $\mathcal{H}^2$ -measure. We look for sufficient conditions on φ which exclude such pathological example.

Let us start with a regularity property of cartesian minimizers of $\mathcal{G}_{\Phi^o}$ for *cylindrical norms over the Euclidean norm*, namely for

$$\Phi(\widehat{\xi}, \xi_{n+1}) = \max(|\widehat{\xi}|, |\xi_{n+1}|),$$

which is exactly the case of the total variation functional.

We need the following regularity result, a special case of [106, Theorem 1].

Theorem 2.46. *Let $\{\widehat{E}_h\}$ be a sequence of minimizers of the Euclidean perimeter in $\widehat{\Omega}$ locally converging to a set $\widehat{E}$ in $\widehat{\Omega}$, and let $x_h \in \partial \widehat{E}_h$ be such that $\lim\limits_{h\to+\infty} x_h = x \in \partial^* \widehat{E}$. Then there exists $\overline{h} \in \mathbb{N}$ such that $x_h \in \partial^* \widehat{E}_h$ for any $h \in \mathbb{N}$, $h \geq \overline{h}$, and $\lim\limits_{h\to+\infty} \nu_{\widehat{E}_h}(x_h) = \nu_{\widehat{E}}(x)$.*

[7] If φ is $\mathcal{C}^\infty$ -uniformly convex norm and $n = 3$, then $\{u \geq \lambda\}$ is smooth [4, Theorem II.7].

Theorem 2.47 (**Local Lipschitz regularity**). *Suppose that* $u \in BV_{\rm loc}(\widehat{\Omega})$ *is a minimizer of the total variation functional*

$$TV(v, \widehat{\Omega}) := \int_{\widehat{\Omega}} |Dv|, \qquad v \in BV_{\rm loc}(\widehat{\Omega}).$$

Then there exists a closed set $\Sigma(u) \subseteq \partial \mathrm{sg}(u)$ *of Hausdorff dimension at most* $n-7$, *with* $\Sigma(u) = \emptyset$ *if* $n \leq 7$, *such that* $\partial \mathrm{sg}(u) \setminus \Sigma(u)$ *is locally Lipschitz.*

Proof. By Proposition 2.36(c) the sets $\{u > \lambda\}$ and $\{u \geq \lambda\}$ are minimizers of the Euclidean perimeter in $\widehat{\Omega}$ for every $\lambda \in \mathbb{R}$. Let $\lambda \in \mathbb{R}$ be such that $\partial\{u > \lambda\}$ (resp. $\partial\{u \geq \lambda\}$) is nonempty. From classical regularity results (see for instance [63, Theorem 11.8] and references therein) it follows that $\partial\{u > \lambda\}$ (resp. $\partial\{u \geq \lambda\}$) *is of class* C^∞ out of a closed set $\partial^{\rm sing}\{u > \lambda\}$ (resp. $\partial^{\rm sing}\{u \geq \lambda\}$) of Hausdorff dimension at most $n-8$. Define

$$\Sigma(u) := \left\{(x, \lambda) \in \partial \mathrm{sg}(u) : \ x \in \partial^{\rm sing}\{u > \lambda\} \text{ or } x \in \partial^{\rm sing}\{u \geq \lambda\}\right\},$$

so that $\Sigma(u)$ has dimension at most $n-7$. From Theorem 2.46 it follows that $\Sigma(u)$ is closed.

Fix

$$(x, \lambda) \in \partial \mathrm{sg}(u) \setminus \Sigma(u). \tag{2.40}$$

One of the following three (not necessarily mutually exclusive) cases holds:

a) $x \in \mathrm{int}(\{u = \lambda\})$;

b) $x \in \partial\{u > \lambda\}$;

c) $x \in \partial\{u \geq \lambda\}$.

In case a) u is locally constant around x, thus, the assertion is immediate.

Assume b). We prove that there exists $r_x > 0$ such that $\partial\{u > \mu\}$ is a graph in direction $\nu_{\{u>\lambda\}}(x)$ for every $\mu \in \mathbb{R}$ such that $\partial\{u > \mu\} \cap B_{r_x}(x) \neq \emptyset$. Indeed, otherwise there would exist $\varepsilon > 0$ and an infinitesimal sequence $(r_h) \subset (0, +\infty)$, and sequences $(\mu_h) \subset \mathbb{R}$, (x_h) with $x_h \in \partial^*\{u > \mu_h\} \cap B_{r_h}(x)$ and

$$|\nu_{\{u>\lambda\}}(x) - \nu_{\{u>\mu_h\}}(x_h)| \geq \varepsilon \qquad \forall h \in \mathbb{N}. \tag{2.41}$$

By Corollary 2.33 u is locally bounded, thus (μ_h) is bounded and we can extract a (not relabelled) subsequence converging to some $\overline{\lambda} \in \mathbb{R}$. There is no loss of generality in assuming (μ_h) nondecreasing. Then $\{u > \mu_h\} \to \{u \geq \overline{\lambda}\}$ in $L^1_{\rm loc}(\widehat{\Omega})$ as $h \to +\infty$. By (2.40) we have $x \in \partial^*\{u \geq \overline{\lambda}\}$, hence from Theorem 2.46 it follows

$$\nu_{\{u>\mu_h\}}(x_h) \to \nu_{\{u\geq\overline{\lambda}\}}(x) \quad \text{as } h \to +\infty. \tag{2.42}$$

Clearly, either $\{u \geq \overline{\lambda}\} \subseteq \{u > \lambda\}$ or $\{u \geq \overline{\lambda}\} \supseteq \{u > \lambda\}$. Since

$$x \in \partial^*\{u > \lambda\} \cap \partial^*\{u \geq \overline{\lambda}\}$$

and $\partial\{u \geq \overline{\lambda}\}$ and $\partial\{u > \lambda\}$ are smooth around x, necessarily

$$\nu_{\{u>\lambda\}}(x) = \nu_{\{u\geq\overline{\lambda}\}}(x).$$

But then from (2.41) and (2.42) we get

$$\varepsilon \leq |\nu_{\{u>\mu_h\}}(x_h) - \nu_{\{u>\lambda\}}(x)| \to 0 \quad \text{as } h \to +\infty,$$

a contradiction.

Thus, for every $x \in \partial^*\{u > \lambda\}$ there exist $r_x > 0$ and $\varepsilon \in (0,1)$ such that for any $\mu \in (\lambda - r_x, \lambda + r_x)$ and $y \in \partial^*\{u > \mu\} \cap B_{r_x}(x)$ one has

$$\nu_{\{u>\lambda\}}(x) \cdot \nu_{\{u>\mu\}}(y) \geq \varepsilon.$$

Notice that for any $(y,\mu) \in \partial^*\mathrm{sg}(u) \setminus \Sigma(u)$ one has that

$$\text{either} \quad \nu_{\mathrm{sg}(u)}(y,\mu) = \frac{(\nu_{\{u>\mu\}}(x), \sigma)}{\sqrt{1+\sigma^2}} \text{ for some } \sigma \geq 0, \quad \text{or} \quad \nu_{\mathrm{sg}(u)}(y,\mu) = e_{n+1}.$$

We want to prove that there exist $\rho > 0$, $\eta \in \mathbb{S}^n$ and $c \in (0,1)$ such that $\mathcal{H}^n$-every $(y,\mu) \in \partial\,\mathrm{sg}(u) \cap B_\rho(x,\lambda)$ there holds

$$\eta \cdot \nu_{\mathrm{sg}(u)}(y,\mu) \geq c, \tag{2.43}$$

so that [93, Lemma 3.10] implies that $\partial\,\mathrm{sg}(u) \cap B_\rho(x,\lambda)$ is a Lipschitz graph in the direction η with Lipschitz constant $L = \sqrt{1/c^2 - 1}$.

Set

$$\rho = r_x, \quad \eta = \frac{1}{\sqrt{2}}\,(\nu_{\{u>\lambda\}}(x), 1).$$

Then for any $(y,\mu) \in \partial^*\mathrm{sg}(u) \cap B_\rho(x,\lambda)$ we have

$$\nu_{\mathrm{sg}(u)}(y,\mu) \cdot \eta = \frac{1}{\sqrt{2}}, \tag{2.44}$$

if $\nu_{\mathrm{sg}(u)}(y,\mu) = e_{n+1}$, and

$$\nu_{\mathrm{sg}(u)}(y,\mu) \cdot \eta = \frac{\nu_{\{u>\mu\}}(y) \cdot \nu_{\{u>\lambda\}}(x) + s}{\sqrt{2}\sqrt{1+s^2}} \geq \frac{\varepsilon + s}{\sqrt{2}\sqrt{1+s^2}} \geq \frac{\varepsilon}{\sqrt{2}}, \tag{2.45}$$

if $y \in \partial\{u > s\}$ (here we use $\frac{a+s}{\sqrt{1+s^2}} \geq a$ for any $a \in (0,1)$ and $s \geq 0$). Formulas (2.44) and (2.45) imply (2.43) with $c = \varepsilon/\sqrt{2}$.

Finally, case c) can be treated as case b).

□

Remark 2.48. The assertion of Theorem 2.47 cannot be improved: if $n \geq 8$, there exists a minimizer u of $\mathcal{G}_{\Phi^o}$ such that the points where $\partial\mathrm{sg}(u)$ is not locally Lipschitz have positive $(n-7)$-dimensional Hausdorff measure. For the Simons cone in $\mathbb{R}^8$ (and with the Euclidean norm), the graph of $u = \chi_C$ cannot be represented as the graph of a Lipschitz function in a neighborhood of the origin.

Theorem 2.47 can be generalized as follows.

Theorem 2.49. *Suppose that* $\Phi : \mathbb{R}^{n+1} \to [0, +\infty)$ *is cylindrical over* φ *with*

$$\varphi^2 \in C^3(\mathbb{R}^n) \quad \text{is uniformly convex.}$$

If u *is a minimizer of* $\mathcal{G}_{\Phi^o}$ *in* $\widehat{\Omega}$, *then* $\partial \mathrm{sg}(u) \setminus \Sigma(u)$ *is locally Lipschitz, where* $\Sigma(u) \subseteq \partial \mathrm{sg}(u)$ *is a closed set of Hausdorff dimension at most* $n-2$ *if* $n > 3$, *and* $\Sigma(u) = \emptyset$ *if* $n = 2, 3$.

Proof. By Proposition 2.36(c) the sets $\{u > \lambda\}$ and $\{u \geq \lambda\}$ are minimizers of P_φ in $\widehat{\Omega}$ for every $\lambda \in \mathbb{R}$. Let $\lambda \in \mathbb{R}$ be such that $\partial\{u > \lambda\}$ (resp. $\partial\{u \geq \lambda\}$) is nonempty. From [4, Theorems II.7] it follows that $\partial\{u > \lambda\}$ (resp. $\partial\{u \geq \lambda\}$) is of class C^2 out of a closed set $\partial^{\mathrm{sing}}\{u > \lambda\}$ (resp. $\partial^{\mathrm{sing}}\{u \geq \lambda\}$) of Hausdorff dimension at most $n-3$. Then

$$\Sigma(u) := \left\{(x, \lambda) \in \partial \mathrm{sg}(u) : \ x \in \partial^{\mathrm{sing}}\{u > \lambda\} \text{ or } x \in \partial^{\mathrm{sing}}\{u \geq \lambda\}\right\},$$

is a set of Hausdorff dimension at most $n-2$. Now we proceed as in the proof of Theorem 2.47 with only difference that closedness of $\Sigma(u)$ and (2.42) follow from [92, Theorem 4.6] instead of Theorem 2.46. □

Remark 2.50. In [93] it is proven that if $n = 2$, B_φ is not a quadrilateral, and u is a minimizer of $\mathcal{G}_{\Phi^o}$ in $\widehat{\Omega}$, then the graph of u is locally Lipschitz around any point of $\partial \mathrm{sg}(u)$.

Remark 2.51. Using the regularity result in [4, Theorem II.8], under the assumption that φ is uniformly convex, smooth and sufficiently close to the Euclidean norm, one can improve Theorem 2.49 by showing that $\Sigma(u)$ has Hausdorff dimension at most $n-5$.

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Chapter 3

Minimizing movements for mean curvature flow of droplets with prescribed contact angle

This chapter is a joint work with G. Bellettini [19] and devoted to study the mean curvature evolutions of capillary droplets on an inhomogeneous hyperplane using the minimizing movements method: we show the existence of a weak evolution, and its compatibility with a distributional solution. We also prove various comparison results. The chapter is organized as follows. In Section 3.1 we study the functional $\mathcal{C}(\cdot,\Omega)$ and its level-set counterpart $\mathtt{C}(\cdot,\Omega)$, including lower semicontinuity and coercivity, which will be useful in Section 3.5. Existence and uniform boundedness results of minimizers of functionals of type $\mathcal{C}_\beta(\cdot,\Omega)+\mathcal{V}$ under suitable hypotheses on $\mathcal{V}$ is established in Section 3.2 from which we deduce the existence and uniform boundedness of minimizers in $BV(\Omega,\{0,1\})$ of $\mathcal{A}(\cdot,E_0,\lambda)$ for any bounded $E_0\in BV(\Omega,\{0,1\})$ and $\|\beta\|_\infty<1$ (Theorem 3.12). The regularity of minimizers are studied in Section 3.4 and Section 3.5 devoted to the comparison principles. In Sections 3.6 - 3.7 we prove the existence of GMM and its coincidence with the distributional solutions to (3)-(4). Finally, in Section 3.8 we prove the short-time existence of a solution to (3)-(4).

Some extra notation in the chapter: for a fixed nonempty $E_0\in BV(\Omega,\{0,1\})$ we define the $L^1(\Omega)$-closed set

$$\mathcal{E}(E_0):=\{E\in BV(\Omega,\{0,1\}):\ E_0\subseteq E\}. \tag{3.1}$$

In this chapter Ω stands for the halfspace $\mathbb{R}^n\times(0,+\infty)$.

3.1 Capillary functionals

Let $\beta\in L^\infty(\partial\Omega)$. The capillary functional $\mathcal{C}_\beta(\cdot,\Omega):BV(\Omega,\{0,1\})\to\mathbb{R}$ and its "level set" version $\mathtt{C}_\beta(\cdot,\Omega):BV(\Omega)\to\mathbb{R}$ are defined as

$$\mathcal{C}_\beta(E,\Omega):=P(E,\Omega)-\int_{\partial\Omega}\beta\,\chi_E\,d\mathcal{H}^n, \tag{3.2}$$

and

$$\mathtt{C}_\beta(u,\Omega) := \int_\Omega |Du| - \int_{\partial\Omega} \beta u d\mathcal{H}^n,$$

respectively. Note that $\mathtt{C}_\beta(\cdot,\Omega)$ is convex, $\mathtt{C}_\beta(u,\Omega) = \mathtt{C}_{-\beta}(-u,\Omega)$ for any $u \in BV(\Omega)$, and $\mathcal{C}_\beta(E,\Omega) = \mathtt{C}_\beta(\chi_E,\Omega)$ for any $E \in BV(\Omega,\{0,1\})$. Moreover, when $\|\beta\|_\infty \le 1$, by (1.4) the functional $\mathcal{C}_\beta(\cdot,\Omega)$ is nonnegative, and the same holds for $\mathtt{C}_\beta(\cdot,\Omega)$ as by (1.9)-(1.10) one has

$$\mathtt{C}_\beta(u,\Omega) = \int_{-\infty}^0 \mathcal{C}_{-\beta}(\{u<t\},\Omega)\,dt + \int_0^{+\infty} \mathcal{C}_\beta(\{u>t\},\Omega)\,dt. \tag{3.3}$$

The functional $\mathtt{C}_\beta(\cdot,\Omega)$ will be useful for the comparison principles (Section 3.5).

Proposition 3.1 (Coercivity of the capillary functionals). *If* $-1 \le \beta \le 1-2\kappa$ $\mathcal{H}^n$*-a.e. on* $\partial\Omega$ *for some* $\kappa \in [0,\frac{1}{2}]$, *then*

$$\kappa P(E) \le \mathcal{C}_\beta(E,\Omega) \le P(E) \qquad \forall E \in BV(\Omega,\{0,1\}). \tag{3.4}$$

Moreover, if $\|\beta\|_\infty \le 1-2\kappa$ *for some* $\kappa \in [0,\frac{1}{2}]$, *then*

$$\kappa \int_{\overline{\Omega}} |Du| \le \mathtt{C}_\beta(u,\Omega) \le \int_{\overline{\Omega}} |Du| \qquad \forall u \in BV(\Omega). \tag{3.5}$$

Proof. The inequality $\kappa P(E) \le \mathcal{C}_\beta(E,\Omega)$ follows from Lemma 1.9 with $A=\Omega$. Moreover, it is immediate to see that

$$\|\beta\|_\infty \le 1 \quad \Longrightarrow \quad \mathcal{C}_\beta(E,\Omega) \le P(E) \qquad \forall E \in BV(\Omega,\{0,1\}). \tag{3.6}$$

Now (3.5) follows from the inequalities

$$\kappa P(\{u<t\},\Omega) + \kappa \int_{\partial\Omega} \chi_{\{u<t\}}\, d\mathcal{H}^n \le \mathcal{C}_{-\beta}(\{u<t\},\Omega) \le P(\{u<t\},\Omega) + \int_{\partial\Omega} \chi_{\{u<t\}}\, d\mathcal{H}^n$$

for a.e. $t<0$ and

$$\kappa P(\{u>t\},\Omega) + \kappa \int_{\partial\Omega} \chi_{\{u>t\}}\, d\mathcal{H}^n \le \mathcal{C}_{\beta}(\{u>t\},\Omega) \le P(\{u>t\},\Omega) + \int_{\partial\Omega} \chi_{\{u>t\}}\, d\mathcal{H}^n$$

for a.e. $t>0$, from (1.9)-(1.10), (3.3) and by [63, Remark 2.14], possibly after extending u to 0 outside Ω. □

Remark 3.2. From the proof of Proposition 3.1 it follows that if $u \ge 0$, then (3.5) holds for any $\beta \in L^\infty(\partial\Omega)$ with $-1 \le \beta \le 1-2\kappa$; if $u \le 0$, (3.5) is valid whenever $-1+2\kappa \le \beta \le 1$.

Remark 3.3. If $\beta > 1$ on a set of infinite $\mathcal{H}^n$-measure, then $\mathcal{C}_\beta(\cdot,\Omega)$ is unbounded from below. Note also that if $\|\beta\|_\infty \le 1$, then $\emptyset$ is the unique minimizer of $\mathcal{C}_\beta(\cdot,\Omega)$ in $BV(\Omega,\{0,1\})$. Indeed, clearly,

$$0 = \mathcal{C}_\beta(\emptyset,\Omega) = \min_{E\in BV(\Omega,\{0,1\})} \mathcal{C}_\beta(E,\Omega).$$

If $E \ne \emptyset$ were a minimizer of $\mathcal{C}_\beta(\cdot,\Omega)$, there would exist $l>0$ such that $|E\setminus\Omega_l|>0$. Now since $\mathrm{Tr}(E) = \mathrm{Tr}(E\cap\overline{\Omega_l})$, by Theorem 1.6 we get

$$0 = \mathcal{C}_\beta(E,\Omega) > \mathcal{C}_\beta(E\cap\overline{\Omega_l},\Omega) \ge 0,$$

a contradiction.

Lemma 3.4 (Lower semicontinuity). *Assume that* $\beta \in L^\infty(\partial\Omega)$*. Then the functionals* $\mathcal{C}_\beta(\cdot,\Omega)$ *and* $\mathtt{C}_\beta(\cdot,\Omega)$ *are* $L^1(\Omega)$*-lower semicontinuous if and only if* $\|\beta\|_\infty \le 1$.

Proof. For $\|\beta\|_\infty \le 1,$ the lower semicontinuity of $\mathcal{C}_\beta(\cdot,\Omega)$ is shown in [29, Lemma 2] and for completeness we reduce it here. By contradiction, suppose that there exist $\varepsilon > 0$ and $E_j, E \in BV(\Omega,\{0,1\})$ such that $E_j \to E$ in $L^1(\Omega)$ and $\mathcal{C}_\beta(E_j) \le \mathcal{C}_\beta(E) - \varepsilon$ or, equivalently,

$$P(E_j,\Omega) \le P(E,\Omega) + \int_{\partial\Omega} \beta[\chi_{E_j} - \chi_E]\, d\mathcal{H}^n - \varepsilon. \tag{3.7}$$

Since $\int_{\partial\Omega} |\chi_E - \chi_{E_j}|\, d\mathcal{H}^n < \infty,$ and $|\beta| \le 1$, there exists $r > 0$ such that

$$\int_{\partial\Omega} \beta[\chi_{E_j} - \chi_E]\, d\mathcal{H}^n \le \int_{B_r\cap\partial\Omega} |\chi_E - \chi_{E_j}|\, d\mathcal{H}^n + \frac{\varepsilon}{2}. \tag{3.8}$$

Set $\Omega_t = \{x \in \Omega : \ \mathrm{distance}(x,\partial\Omega) < t\}.$ We have (see [63, proof of Proposition 2.6]):

$$\int_{\hat{B}_r} |\chi_E - \chi_{E_j}|\, d\mathcal{H}^n \le P(E,\Omega_t) + P(E_j,\Omega_t) + C(t)\int_{\Omega_t} |\chi_E - \chi_{E_j}|\, dx.$$

Hence for small $t > 0$ from (3.7) and (3.8) we get

$$P(E_j,\Omega) \le P(E,\Omega) + P(E,\Omega_t) + P(E_j,\Omega_t) + C(t)\int_{\Omega_t} |\chi_E - \chi_{E_j}|\, dx - \frac{\varepsilon}{2},$$

i.e.

$$P(E_j,\Omega\setminus\Omega_t) \le P(E,\Omega) + P(E,\Omega_t) + C(t)\int_{\Omega_t} |\chi_E - \chi_{E_j}|\, dx - \frac{\varepsilon}{2}.$$

Whence taking $\liminf$ as $j \to +\infty$ from the lower semicontinuity of perimeter we get

$$P(E,\Omega\setminus\Omega_t) \le P(E,\Omega) + P(E,\Omega_t) - \frac{\varepsilon}{2},$$

thus,

$$\frac{\varepsilon}{4} \le P(E,\Omega_t).$$

Now letting $t \to 0+$ we get $\varepsilon = 0,$ a contradiction. The lower semicontinuity of $\mathtt{C}_\beta(\cdot,\Omega)$ follows easily from the lower semicontinuity of $\mathcal{C}_\beta(\cdot,\Omega)$, (3.3) and Fatou's Lemma.

Assume now that $\|\beta\|_\infty > 1,$ i.e. the set $\{\hat{x} \in \partial\Omega : \ |\beta(\hat{x})| > 1\}$ has positive $\mathcal{H}^n$-measure. Let for some $\varepsilon, \delta_0 > 0$ the set $\hat{A} := \{\beta > 1 + \varepsilon\}$ satisfy $|\hat{A}| \ge \delta_0$. By Lusin's theorem, for any $k > \frac{4\|\beta\|_\infty}{\varepsilon\delta_0}$ there exists $\beta_k \in C(\partial\Omega)$ such that $\mathcal{H}^n(\{\beta \ne \beta_k\}) < \frac{1}{k}$ and $\|\beta_k\|_\infty \le \|\beta\|_\infty$. Let k be so large that $\mathcal{H}^n(\{\beta_k > 1+\varepsilon\}) \ge \delta_0/2$ and choose an open set $\hat{O} \subset \{\beta_k > 1+\varepsilon\}$ of finite perimeter such that $\delta_0/4 \le \mathcal{H}^n(\hat{O}) < +\infty.$ Define the sequence of sets $E_m := \hat{O} \times (0, \frac{1}{m}) \subset \Omega.$ Clearly, $E_m \to \emptyset$ in $L^1(\Omega)$ as $m \to +\infty.$ Then, indicating by $P(\hat{O})$ the perimeter of $\hat{O}$ in $\mathbb{R}^n,$ from the relations

$$\begin{aligned}\mathcal{C}_\beta(E_m,\Omega) =& \frac{1}{m}P(\hat{O}) + \mathcal{H}^n(\hat{O}) - \int_{\hat{O}} \beta d\mathcal{H}^n \\ \le& \frac{1}{m}P(\hat{O}) + \mathcal{H}^n(\hat{O}) - \int_{\hat{O}} \beta_k d\mathcal{H}^n + \int_{\hat{O}} |\beta - \beta_k| d\mathcal{H}^n \\ \le& \frac{1}{m}P(\hat{O}) - \varepsilon\mathcal{H}^n(\hat{O}) + 2\|\beta\|_\infty \mathcal{H}^n(\hat{O}\cap\{\beta \ne \beta_k\}) \le \frac{1}{m}P(\hat{O}) - \frac{\varepsilon\delta_0}{4},\end{aligned}$$

we establish

$$\liminf_{m\to+\infty} \mathcal{C}_\beta(E_m,\Omega) \le -\frac{\varepsilon\delta_0}{4} < 0 = \mathcal{C}_\beta(\emptyset,\Omega).$$

Since $\mathtt{C}_\beta(\chi_E,\Omega) = \mathcal{C}_\beta(E,\Omega)$, one has also $\liminf_{m\to+\infty} \mathtt{C}_\beta(\chi_{E_m},\Omega) < 0 = \mathtt{C}_\beta(0,\Omega)$. Hence $\mathcal{C}_\beta(\cdot,\Omega)$ and $\mathtt{C}_\beta(\cdot,\Omega)$ are not $L^1(\Omega)$-lower semicontinuous.

Finally, the case when $\{\beta < -1-\varepsilon\}$ has positive Lebesgue measure can be treated in a similar way. □

Remark 3.5. If Ω is an arbitrary bounded open set with Lipschitz boundary and $\|\beta\|_\infty \le 1$, then the lower semicontinuity of $\mathcal{C}_\beta(\cdot,\Omega)$ is a consequence of [7, Theorem 3.4]. In this case $\mathcal{C}_\beta(\cdot,\Omega)$ is bounded from below by $-\mathcal{H}^n(\partial\Omega)$. Hence again Fatou's lemma and (3.3) yield lower semicontinuity of $\mathtt{C}_\beta(\cdot,\Omega)$.

3.2 Existence of minimizers for some functionals

In this section we prove an existence result for minimum problems of type

$$\inf_{E\in BV(\Omega,\{0,1\})} \mathcal{G}_\beta(E), \qquad \mathcal{G}_\beta(E) := \mathcal{C}_\beta(E,\Omega) + \mathcal{V}(E), \tag{3.9}$$

where $\mathcal{V} : BV(\Omega,\{0,1\}) \to (-\infty,+\infty]$. Since $\mathcal{C}_\beta(\cdot,\Omega)$ is finite in $BV(\Omega,\{0,1\})$, the functional $\mathcal{G}_\beta$ is well-defined in $BV(\Omega,\{0,1\})$. We study (3.9) under the following hypotheses on $\mathcal{V}$:

Hypothesis 3.6. (a) $\mathcal{V}$ is bounded from below in $BV(\Omega,\{0,1\})$ and there exists a cylinder $C_r^K \subset \Omega$, $K > 1$ such that $\mathcal{V}(C_r^K) < +\infty$;

(b) $\mathcal{V}(E) \ge \mathcal{V}(E\cap C_\rho^l)$ for any $E \in BV(\Omega,\{0,1\})$, $\rho\in(r,+\infty]$, and $l \in (K-1,K+1)$;

(c) $\mathcal{V}(E) \ge \mathcal{V}(E\setminus(C_{\rho_1}^K \setminus \overline{C_{\rho_2}^K}))$ for any $E\in BV(\Omega,\{0,1\})$ and $r<\rho_2<\rho_1<+\infty$;

(d) $\mathcal{V}$ is $L^1(\Omega)$-lower semicontinuous in $BV(\Omega,\{0,1\})$.

Example 3.7. Besides (3.35) the following functionals $\mathcal{V} : BV(\Omega,\{0,1\}) \to (-\infty,+\infty]$ satisfy Hypothesis 3.6:

1) given $f\in L^1_{\mathrm{loc}}(\Omega)$ with $f\ge 0$ a.e. in $\Omega\setminus C_r^l$ for some $r,l>0$,

$$\mathcal{V}(E) = \int_E f\,dx.$$

In particular, we may take $f = \lambda\tilde{d}_{E_0}$ with $\emptyset \ne E_0 \in BV(\Omega,\{0,1\})$ and $E_0 \subset C_r^h$ so that by (3.29) $\mathcal{G}_\beta$ coincides with $\mathcal{A}_\beta(\cdot,E_0,\lambda) + \int_{E_0}\tilde{d}_{E_0}dx$.

2) Given a bounded set $E_0\in BV(\Omega,\{0,1\})$, $\mathcal{V}(E) = |E\Delta E_0|^p$, $p>0$.

Given $\mathcal{V}$ satisfying Hypothesis 3.6 set

$$\mathfrak{a} := \kappa^{-1}\left(\sup_{R>r}\inf_{E\in BV(C_R^K,\{0,1\})}\mathcal{G}_\beta(E) - \inf\mathcal{V}\right).$$

Clearly, $\kappa\mathfrak{a} \le \mathcal{G}_\beta(C_r^K) - \inf\mathcal{V}$, hence $\inf\mathcal{G}_\beta < +\infty$.

Theorem 3.8 (Existence of minimizers and uniform bound). *Suppose that Hypothesis 3.6 holds. Suppose also $\beta\in L^\infty(\partial\Omega)$ and there exists $\kappa\in(0,\frac12]$ such that $-1\le\beta\le 1-2\kappa$ $\mathcal{H}^n$-a.e on $\partial\Omega$. Then the minimum problem*

$$\inf_{E\in BV(\Omega,\{0,1\})}\mathcal{G}_\beta(E)$$

has a solution. Moreover, any minimizer is contained in $C^K_{\mathcal{R}_0}$, where[1]

$$\mathcal{R}_0 := r+1+\max\left\{8^{n^2+n+1}\,\mathfrak{a}^{\frac{n+1}{n}},\,4\mu(\kappa,n)\right\} \tag{3.10}$$

and $\mu(\kappa,n) := (1/\kappa+2)^{\frac{n+1}{n}}$.

Remark 3.9. In case of Example 3.7 1) with $f=\lambda\tilde d_{E_0}$ for some $C_r^K\supseteq E_0$,

$$\kappa\mathfrak{a} \le \kappa\sup_{R>r}\inf_{E\in BV(C_R^K,\{0,1\})}\mathcal{A}_\beta(E,E_0,\lambda) \le \kappa\mathcal{A}_\beta(E_0,E_0,\lambda) = \kappa\mathcal{C}_\beta(E_0,\Omega) \le \kappa P(E_0).$$

Hence, $\mathcal{R}_0\le R_0$, where R_0 is defined in (3.31). The same is true if $\mathcal{V}$ is as in (3.35).

The assumption on β and the $L^1(\Omega)$-lower semicontinuity of $\mathcal{C}_\beta(\cdot,\Omega)$ (Lemma 3.4) imply the $L^1(\Omega)$-lower semicontinuity of $\mathcal{G}_\beta$. Moreover, the coercivity (3.4) of $\mathcal{C}_\beta(\cdot,\Omega)$, Hypothesis 3.6 (a) and (3.6) imply the coercivity of $\mathcal{G}_\beta$:

$$\mathcal{G}_\beta(E) \ge \kappa P(E) + \inf\mathcal{V} \qquad \forall E\in BV(\Omega,\{0,1\}). \tag{3.11}$$

The main problem in the proof of existence of minimizers of $\mathcal{G}_\beta$ is the lack of compactness due to the unboundedness of Ω. However, for every $R>0$ inequality (3.11), the compactness theorem in $BV(C_R^K,\{0,1\})$ (see for instance [11, Theorems 3.23 and 3.39]) and the lower semicontinuity of $\mathcal{G}_\beta$ imply that there exists a solution $E^R\in BV(C_R^K,\{0,1\})$ of

$$\inf_{E\in BV(C_R^K,\{0,1\})}\mathcal{G}_\beta(E).$$

To prove Theorem 3.8 we mainly follow [29, Section 4], where the existence of volume preserving minimizers of $\mathcal{C}_\beta(\cdot,\Omega)$ has been shown. We need two preliminary lemmas. As in [29, Section 3] first we show that one can choose a minimizing sequence consisting of bounded sets.

Lemma 3.10 (Truncations with horizontal hyperplanes and vertical cylinders). *Suppose that Hypothesis 3.6 holds. Then*

$$\inf_{E\in BV(\Omega,\{0,1\})}\mathcal{G}_\beta(E) = \inf_{R>0}\inf_{E\in BV(C_R^K,\{0,1\})}\mathcal{G}_\beta(E). \tag{3.12}$$

[1] One could refine the expression of $\mathcal{R}_0$ using the isoperimetric inequality (Remark 1.1 C)), but we do not need this here.

Proof. We need two intermediate steps. The first step concerns truncations with horizontal hyperplanes.

Step 1. We have

$$\inf_{E\in BV(\Omega,\{0,1\})} \mathcal{G}_\beta(E) = \inf_{E\in BV(\Omega_K,\{0,1\})} \mathcal{G}_\beta(E).$$

Indeed, it suffices to show that if $E\setminus \Omega_{K-\frac{1}{4}} \neq \emptyset,$ then

$$\mathcal{G}_\beta(E) \geq \mathcal{G}_\beta\big(E\cap \overline{\Omega_{K-\frac{1}{2}}}\big).$$

Clearly, E and $E\cap\overline{\Omega_{K-\frac{1}{2}}}$ have the same trace on $\partial\Omega$ and thus

$$\int_{\partial\Omega}[1+\beta]\,\chi_E\, d\mathcal{H}^n = \int_{\partial\Omega}[1+\beta]\,\chi_{E\cap\overline{\Omega_{K-\frac{1}{2}}}}\, d\mathcal{H}^n.$$

From Theorem 1.6 we have

$$P(E) > P\big(E\cap\overline{\Omega_{K-\frac{1}{2}}}\big).$$

By Hypothesis 3.6 (b) we have also

$$\mathcal{V}(E) \geq \mathcal{V}(E\cap\overline{\Omega_{K-\frac{1}{2}}}),$$

therefore from the definition of $\mathcal{G}_\beta$ we get even the strict inequality

$$\mathcal{G}_\beta(E) > \mathcal{G}_\beta(E\cap\overline{\Omega_{K-\frac{1}{2}}}). \tag{3.13}$$

The second step is more delicate and concerns truncations with the lateral boundary of vertical cylinders.

Step 2. For any $\varepsilon\in(0,1)$ there exists $R_\varepsilon > r$ and $E_\varepsilon\in BV(C^K_{R_\varepsilon},\{0,1\})$ such that

$$\mathcal{G}_\beta(E_\varepsilon) \leq \inf_{E\in BV(\Omega_K,\{0,1\})} \mathcal{G}_\beta(E)+\varepsilon.$$

Indeed, according to Step 1 and Hypothesis 3.6 (a), given $\varepsilon > 0$ there exists $F_\varepsilon \in BV(\Omega_K,\{0,1\})$ with $F_\varepsilon\subset\overline{\Omega_{K-\frac{1}{4}}}$ such that

$$\mathcal{G}_\beta(F_\varepsilon) < \inf_{E\in BV(\Omega,\{0,1\})} \mathcal{G}_\beta(E)+\frac{\varepsilon}{2} < +\infty.$$

Since $|F_\varepsilon| < +\infty,$ for sufficiently large $R > r$ one has

$$|F_\varepsilon\cap(C^K_{R+1}\setminus C^K_R)| = \int_R^{R+1}\mathcal{H}^n(F_\varepsilon\cap\partial C^K_\rho)\,d\rho < \frac{\varepsilon}{2}.$$

Hence there exists $R_\varepsilon\in(R,R+1)$ such that

$$\mathcal{H}^n\big(F_\varepsilon\cap\partial C^K_{R_\varepsilon}\big)\leq\frac{\varepsilon}{2},\qquad \mathcal{H}^n\big(\Omega\cap\partial^* F_\varepsilon\cap\partial C^K_{R_\varepsilon}\big)=0.$$

Now, let $E_\varepsilon := F_\varepsilon\cap C^K_{R_\varepsilon}.$ Since $\mathcal{H}^n\big(\Omega\cap\partial^*F_\varepsilon\cap\partial C^K_{R_\varepsilon}\big)=0,$ we have

$$\begin{aligned} P(E_\varepsilon,\Omega) =& P(E_\varepsilon,\Omega_K) = P(F_\varepsilon,\Omega_K)+\mathcal{H}^n\big(F_\varepsilon\cap\partial C^K_{R_\varepsilon}\big) - P\big(F_\varepsilon,\Omega_K\setminus\overline{C^K_{R_\varepsilon}}\big)\\ =& P(F_\varepsilon,\Omega)+\mathcal{H}^n\big(F_\varepsilon\cap\partial C^K_{R_\varepsilon}\big) - P\big(F_\varepsilon,\Omega_K\setminus\overline{C^K_{R_\varepsilon}}\big). \end{aligned} \tag{3.14}$$

By Hypothesis 3.6 (a), $\mathcal{V}(F_\varepsilon) \geq \mathcal{V}(E_\varepsilon),$ thus employing (3.14) we get

$$\mathcal{G}_\beta(F_\varepsilon) \geq \mathcal{G}_\beta(E_\varepsilon) - \mathcal{H}^n(F_\varepsilon \cap \partial C^K_{R_\varepsilon}) + P(F_\varepsilon, \Omega_K \setminus \overline{C^K_{R_\varepsilon}}) - \int_{\partial\Omega} \beta\, \chi_{F_\varepsilon \setminus C^K_{R_\varepsilon}}\, d\mathcal{H}^n.$$

By Lemma 1.9 applied with $E = F_\varepsilon$ and $A = \Omega_K \setminus \overline{C^K_{R_\varepsilon}},$ we have

$$P(F_\varepsilon, \Omega_K \setminus \overline{C^K_{R_\varepsilon}}) - \int_{\partial\Omega} \beta\, \chi_{F_\varepsilon \setminus C^K_{R_\varepsilon}}\, d\mathcal{H}^n \geq 0.$$

Consequently, from the choice of F_ε and R_ε we get

$$\mathcal{G}_\beta(E_\varepsilon) \leq \mathcal{G}_\beta(F_\varepsilon) + \mathcal{H}^n(F_\varepsilon \cap \partial C^K_{R_\varepsilon}) < \inf_{E \in BV(\Omega,\{0,1\})} \mathcal{G}_\beta(E) + \varepsilon.$$

This concludes the proof of Step 2.

Now, observe that

$$\inf_{E \in BV(\Omega,\{0,1\})} \mathcal{G}_\beta(E) \leq \inf_{R>0} \inf_{E \in BV(C^K_R,\{0,1\})} \mathcal{G}_\beta(E).$$

On the other hand, since the mapping

$$R \in (0,+\infty) \mapsto \inf_{E \in BV(C^K_R,\{0,1\})} \mathcal{G}_\beta(E)$$

is nonincreasing, Step 2 implies

$$\inf_{E \in BV(\Omega,\{0,1\})} \mathcal{G}_\beta(E) \geq \inf_{R>0} \inf_{E \in BV(C^K_R,\{0,1\})} \mathcal{G}_\beta(E),$$

therefore (3.12) follows. □

As in [29, Lemma 3] the following lemma holds.

Lemma 3.11 (Good choice of a radius). *Suppose that β satisfies (3.30) and Hypothesis 3.6 holds. Let E^R be a minimizer of $\mathcal{G}_\beta$ in $BV(C^K_R,\{0,1\})$. Then for any $R > \mathcal{R}_0$ there exists $t_R \in [r+1, \mathcal{R}_0]$ such that*

$$\mathcal{H}^n(E^R \cap \partial C^K_{t_R}) = 0.$$

Hence

$$P(E^R, \Omega) = P\big(E^R \setminus \overline{C^K_{t_R}}, \Omega\big) + P\big(E^R \cap C^K_{t_R}, \Omega\big). \tag{3.15}$$

Proof. The idea of the proof is to cut the E^R with vertical cylinders, similarly to [29, Lemma 5] where cuts with horizontal hyperplanes are performed.

For $R > \mathcal{R}_0$ by the isoperimetric-type inequality [44, Theorem VI], (3.11), the minimality of E^R and by the definition of $\mathfrak{a}$ we have

$$|E^R|^{\frac{n}{n+1}} \leq P(E^R) \leq \frac{\mathcal{G}_\beta(E^R) - \inf \mathcal{V}}{\kappa} = \frac{1}{\kappa}\left(\inf_{E \in BV(C^K_R,\{0,1\})} \mathcal{G}_\beta(E) + \inf \mathcal{V}\right) \leq \mathfrak{a}.$$

Thus, for any $0 < a < b$ one has

$$|E^R \cap (C_b^K \setminus C_a^K)| \le \mathfrak{a}^{\frac{n+1}{n}}. \tag{3.16}$$

Take $r + 1 < r_1 < r_2 < r_3 < \mathcal{R}_0$ such that

$$\mathcal{H}^n(\Omega \cap \partial^* E^R \cap \partial C_{r_i}^K) = 0, \quad i = 1, 2, 3,$$

and set

$$v_1 = |E^R \cap (C_{r_2}^K \setminus C_{r_1}^K)|, \qquad v_2 = |E^R \cap (C_{r_3}^K \setminus C_{r_2}^K)|,$$
$$m = \max_{i=1,2,3} \mathcal{H}^n(E^R \cap \partial C_{r_i}^K).$$

Step 1. We claim that

$$\min\{v_1, v_2\} \le \mu m^{\frac{n+1}{n}}, \tag{3.17}$$

where $\mu := \mu(\kappa, n) > 0$.

It suffices to prove that

$$v_1^{\frac{n}{n+1}} + v_2^{\frac{n}{n+1}} \le 2\mu^{\frac{n}{n+1}} m.$$

We have

$$\begin{aligned} v_1^{\frac{n}{n+1}} \le & P\big(E^R \cap (C_{r_2}^K \setminus \overline{C_{r_1}^K})\big) \le P(E^R, C_{r_2}^K \setminus \overline{C_{r_1}^K}) + \mathcal{H}^n(E^R \cap \partial C_{r_1}^K) \\ & + \mathcal{H}^n(E^R \cap \partial C_{r_2}^K) + \int_{\partial\Omega} \chi_{E^R \cap (C_{r_2}^K \setminus \overline{C_{r_1}^K})}\, d\mathcal{H}^n \\ \le & P(E^R, C_{r_2}^K \setminus \overline{C_{r_1}^K}) + \int_{\partial\Omega} \chi_{E^R \cap (C_{r_2}^K \setminus \overline{C_{r_1}^K})}\, d\mathcal{H}^n + 2m. \end{aligned}$$

Similarly,

$$v_2^{\frac{n}{n+1}} \le P(E^R, C_{r_3}^K \setminus \overline{C_{r_2}^K}) + \int_{\partial\Omega} \chi_{E^R \cap (C_{r_3}^K \setminus \overline{C_{r_2}^K})}\, d\mathcal{H}^n + 2m.$$

Hence

$$v_1^{\frac{n}{n+1}} + v_2^{\frac{n}{n+1}} \le P(E^R, C_{r_3}^K \setminus \overline{C_{r_1}^K}) + \int_{\partial\Omega} \chi_{E^R \cap (C_{r_3}^K \setminus \overline{C_{r_1}^K})}\, d\mathcal{H}^n + 4m. \tag{3.18}$$

Comparing $E^R \setminus (C_{r_3}^K \setminus \overline{C_{r_1}^K})$ with E^R, we get $\mathcal{G}_\beta(E^R) \le \mathcal{G}_\beta(E^R \setminus (C_{r_3}^K \setminus \overline{C_{r_1}^K}))$, therefore from Hypothesis 3.6 (c) we obtain

$$P(E^R) \le P\big(E^R \setminus (C_{r_3}^K \setminus \overline{C_{r_1}^K})\big) + \int_{\partial\Omega} [1 + \beta]\, \chi_{E^R \cap (C_{r_3}^K \setminus \overline{C_{r_1}^K})}\, d\mathcal{H}^n. \tag{3.19}$$

Inserting in (3.19) the identity

$$\begin{aligned} P(E^R \setminus (C_{r_3}^K \setminus \overline{C_{r_1}^K})) = & P(E^R) + \mathcal{H}^n(E^R \cap \partial C_{r_1}^K) + \mathcal{H}^n(E^R \cap \partial C_{r_3}^K) \\ & - P(E^R, C_{r_3}^K \setminus \overline{C_{r_1}^K}) - \int_{\partial\Omega} \chi_{E^R \cap (C_{r_3}^K \setminus \overline{C_{r_1}^K})}\, d\mathcal{H}^n, \end{aligned}$$

we get

$$P(E^R, C^K_{r_3} \setminus \overline{C^K_{r_1}}) - \int_{\partial\Omega} \beta\, \chi_{E^R \cap (C^K_{r_3} \setminus \overline{C^K_{r_1}})}\, d\mathcal{H}^n \le 2m. \tag{3.20}$$

By Lemma 1.9 applied with $A = C^K_{r_3} \setminus \overline{C^K_{r_1}}$ and $E = E^R,$ the left-hand-side of (3.20) is not less than

$$\kappa P(E^R, C^K_{r_3} \setminus \overline{C^K_{r_1}}) + \kappa \int_{\partial\Omega} \chi_{E^R \cap (C^K_{r_3} \setminus \overline{C^K_{r_1}})}\, d\mathcal{H}^n,$$

hence

$$P\big(E^R, C^K_{r_3} \setminus \overline{C^K_{r_1}}\big) + \int_{\partial\Omega} \chi_{E^R \cap (C^K_{r_3} \setminus \overline{C^K_{r_1}})}\, d\mathcal{H}^n \le \frac{2m}{\kappa}.$$

Then from (3.18) it follows that

$$v_1^{\frac{n}{n+1}} + v_2^{\frac{n}{n+1}} \le \left(\frac{2m}{\kappa} + 4m\right) = 2\mu^{\frac{n}{n+1}} m.$$

This finishes the proof of Step 1.

Before going to Step 2 we need some preliminaries. Choose any $R \ge \mathcal{R}_0.$ Let $a_0 = r+1,$ $b_0 = \mathcal{R}_0.$ Given $r+1 \le a_k \le b_k \le \mathcal{R}_0,\ \ k \in \mathbb{N},$ define

$$v_k = |E^R \cap (C^K_{b_k} \setminus C^K_{a_k})|.$$

By (3.13) $E^R \setminus \Omega_{K-\frac14} = \emptyset,$ hence

$$|E^R \cap (C^K_b \setminus C^K_a)| = \int_a^b \mathcal{H}^n(E^R \cap \partial C^K_\rho)\, d\rho, \quad 0 \le a < b.$$

Therefore, for $h_k = \frac{b_k - a_k}{4}$ it is possible to find $r_{k,1} \in (a_k, a_k + h_k),\ r_{k,2} \in (\frac{a_k+b_k}{2} - \frac{h_k}{2}, \frac{a_k+b_k}{2} + \frac{h_k}{2})$ and $r_{k,3} \in (b_k - h_k, b_k)$ such that

$$\mathcal{H}^n(E^R \cap \partial C^K_{r_{k,i}}) \le \frac{v_k}{h_k}, \quad \mathcal{H}^n(\Omega \cap \partial^* E^R \cap \partial C^K_{r_{k,i}}) = 0 \quad \text{for } i = 1,2,3\,. \tag{3.21}$$

We choose

$$(a_{k+1}, b_{k+1}) = \begin{cases} (r_{k,1}, r_{k,2}) & \text{if } |E^R \cap (C^K_{r_{k,1}} \setminus C^K_{r_{k,2}})| \le |E^R \cap (C^K_{r_{k,2}} \setminus C^K_{r_{k,3}})|, \\ (r_{k,2}, r_{k,3}) & \text{if } |E^R \cap (C^K_{r_{k,1}} \setminus C^K_{r_{k,2}})| > |E^R \cap (C^K_{r_{k,2}} \setminus C^K_{r_{k,3}})|. \end{cases}$$

Let

$$m_k = \max_{i=1,2,3} \mathcal{H}^n(E^R \cap \partial C^K_{r_{k,i}}).$$

Step 2. Using the definition of $\mathcal{R}_0$ we show that

$$m_k \le \left(\frac12\right)^{(\frac{n+1}{n})^k}. \tag{3.22}$$

Indeed, according to (3.17), (3.21) and the definition of (a_k, b_k) one has

$$v_{k+1} \le \mu m_k^{\frac{n+1}{n}}, \qquad m_k \le \frac{v_k}{h_k}.$$

By construction, $b_{k+1} - a_{k+1} \ge \frac{b_k - a_k}{8}$, i.e. $h_{k+1} \ge \frac{h_k}{8}$. By induction one can check that

$$m_k \le \left(8^{\sum\limits_{j=1}^{k} j\alpha^j} \left(\frac{\mu}{h_0}\right)^{\sum\limits_{j=1}^{k} \alpha^j} \frac{v_0}{h_0} \right)^{1/\alpha^k}, \tag{3.23}$$

where $\alpha := \frac{n}{n+1}$. Note that

$$\sum_{j=1}^{k} j\alpha^j \le \alpha \sum_{j\ge 1} j\alpha^{j-1} = \frac{\alpha}{(1-\alpha)^2} = n(n+1).$$

Since $h_0 = \frac{\mathcal{R}_0 - r - 1}{4}$ and $v_0 \le \mathfrak{a}^{\frac{n+1}{n}}$ by (3.16), the choice of $\mathcal{R}_0$ in (3.10) implies $8^{n(n+1)}\, v_0/h_0 \le 1/2$. Moreover $\left(\frac{\mu}{h_0}\right)^{\sum\limits_{j=1}^{k}\alpha^j} \le 1$, since $\frac{\mu}{h_0} = \frac{4\mu}{R_0 - r - 1} \le 1$. Now (3.22) follows from these estimates and (3.23).

Step 3. Let $i_k \in \{1,2,3\}$ be such that $m_k = \mathcal{H}^n(E^R \cap \partial C^K_{r_{k,i_k}})$. Since $a_k \le r_{k,i_k} \le b_k$, $\{a_k\}$ is nondecreasing and $\{b_k\}$ is nonincreasing, there exists $t_R \in [r+1, \mathcal{R}_0]$ such that $r_{k,i_k} \to t_R$ (possibly up to a subsequence). Then, by Step 2,

$$\mathcal{H}^n(E^R \cap \partial C^K_{t_R}) = \lim_{k\to+\infty} m_k = 0,$$

which concludes the proof of the lemma. □

Proof of Theorem 3.8. Let us prove the existence of a minimizer of $\mathcal{G}_\beta$. For $R > \mathcal{R}_0$ let $t_R \in [r+1, \mathcal{R}_0]$ be as in Lemma 3.11. Then from (3.15) and $\mathcal{V}(E^R) \ge \mathcal{V}(E^R \cap C^K_{t_R})$ we get

$$\mathcal{G}_\beta(E^R) \ge \mathcal{G}_\beta(E^R \cap C^K_{t_R}) + P\big(E^R \setminus \overline{C^K_{t_R}}, \Omega\big) - \int_{\partial\Omega} \beta \chi_{E^R \setminus \overline{C^K_{t_R}}} d\mathcal{H}^n. \tag{3.24}$$

By (3.4) and the isoperimetric-type inequality

$$P\big(E^R \setminus \overline{C^K_{t_R}}, \Omega\big) - \int_{\partial\Omega} \beta\, \chi_{E^R \setminus \overline{C^K_{t_R}}}\, d\mathcal{H}^n \ge \kappa P\big(E^R \setminus \overline{C^K_{t_R}}\big) \ge \kappa \big|E^R \setminus \overline{C^K_{t_R}}\big|^{\frac{n}{n+1}}. \tag{3.25}$$

Thus from (3.24)

$$\mathcal{G}_\beta(E^R) \ge \mathcal{G}_\beta(E^R \cap C^K_{t_R}).$$

Hence, $F^R := E^R \cap C^K_{t_R} \subseteq C^K_{\mathcal{R}_0}$ satisfies

$$\min_{E \in BV(C^K_R, \{0,1\})} \mathcal{G}_\beta(E) = \mathcal{G}_\beta(F^R).$$

From (3.4) and the minimality of F^R we get

$$\kappa P(F^R) \le \mathcal{C}_\beta(F^R,\Omega) \le \mathcal{G}_\beta(F^R) - \inf \mathcal{V} \le \kappa\mathfrak{a},$$

and thus, by compactness there exists $E \in BV(C^K_{\mathcal{R}_0},\{0,1\})$ such that (up to a subsequence) $F^R \to E$ in $L^1(\Omega)$ as $R\to+\infty$. From the $L^1(\Omega)$-lower semicontinuity of $\mathcal{G}_\beta$ and from (3.12) we conclude that E is a minimizer of $\mathcal{G}_\beta$.

Now we prove that any minimizer E of $\mathcal{G}_\beta$ satisfies $E\subseteq C^K_{\mathcal{R}_0}$. Arguing as in the proof of (3.13) one can show that $E\subseteq \overline{\Omega_{K-\frac14}}$.

Claim. There exists $R>r+1$ (possibly depending on $\mathcal{V}$ and r) such that $E\subseteq C^K_R$.

For any $\rho>1$ such that $\mathcal{H}^n(\Omega\cap\partial^*E\cap\partial C^K_\rho)=0$, by the minimality of E we have $\mathcal{G}_\beta(E)\le\mathcal{G}_\beta(E\cap C^K_\rho)$, i.e.

$$P(E,\Omega_K\setminus\overline{C^K_\rho}) - \int_{\partial\Omega}\beta\chi_{E\setminus C^K_\rho}\,d\mathcal{H}^n \le \mathcal{H}^n(E\cap\partial C^K_\rho). \tag{3.26}$$

By Lemma 1.9

$$P(E,\Omega_K\setminus\overline{C^K_\rho}) - \int_{\partial\Omega}\beta\chi_{E\setminus C^K_\rho}\,d\mathcal{H}^n \ge \kappa\Big(P(E,\Omega_K\setminus\overline{C^K_\rho}) + \int_{\partial\Omega}\chi_{E\setminus C^K_\rho}\,d\mathcal{H}^n\Big). \tag{3.27}$$

Moreover, by the isoperimetric-type inequality,

$$|E\setminus C^K_\rho|^{\frac{n}{n+1}} \le P(E,\Omega_K\setminus\overline{C^K_\rho}) + \mathcal{H}^n(E\cap\partial C^K_\rho) + \int_{\partial\Omega}\chi_{E\setminus C^K_\rho E}\,d\mathcal{H}^n.$$

therefore, (3.26) and (3.27) imply

$$|E\setminus C^K_\rho|^{\frac{n}{n+1}} \le \frac{\kappa+1}{\kappa}\,\mathcal{H}^n(E\cap\partial C^K_\rho). \tag{3.28}$$

Set $m(\rho)=|E\setminus C^K_\rho|$. Clearly, $m:(1,+\infty)\to[0,|E|]$. Moreover, m is absolutely continuous, nonincreasing, $\lim\limits_{\rho\to+\infty} m(\rho)=0$ and $\mathcal{H}^n(E\cap\partial C^K_\rho)=-m'(\rho)$ for a.e. $\rho>r+1$. By (3.28) $-m'(\rho)\ge\frac{\kappa+1}{\kappa}(n+1)m(\rho)^{\frac{n}{n+1}}$. If E is unbounded, then $m(\rho)>0$ for any $\rho>r+1$, and thus, for any $\rho_1,\rho_2>r+1$, $\rho_1<\rho_2$ we have

$$m(\rho_1)^{\frac{1}{n+1}} - m(\rho_2)^{\frac{1}{n+1}} \ge \frac{\kappa+1}{\kappa}(\rho_2-\rho_1).$$

Now letting $\rho_2\to+\infty$ we obtain $m(\rho_1)=+\infty$, a contradiction. Consequently, there exists $R>r+1$ such that $m(R)=0$, i.e. $E\subseteq C^K_R$.

From the claim it follows that E is a minimizer of $\mathcal{G}_\beta$ also in $BV(C^K_R,\{0,1\})$. By Lemma 3.11 we can find $t_R\in[r+1,\mathcal{R}_0]$ such that $\mathcal{H}^n(E\cap\partial C^K_{t_R})=0$. Then using $\mathcal{V}(E)\ge\mathcal{V}(E\cap C^K_{t_R})$, the relations (3.24) - (3.25) applied with E in place of E^R imply

$$\mathcal{G}_\beta(E)\ge\mathcal{G}_\beta(E\cap C^K_{t_R}) + \kappa\big|E\setminus\overline{C^K_{t_R}}\big|^{\frac{n}{n+1}}.$$

Therefore, the minimality of E yields $\big|E\setminus\overline{C^K_{t_R}}\big|=0$, i.e. $E\subseteq C^K_{t_R}$. Since $t_R\le\mathcal{R}_0$, the conclusion follows. $\square$

3.3 Capillary Almgren-Taylor-Wang-type functional

In the sequel, for a given nonempty set $F \subseteq \Omega$, d_F stands for the distance function from the boundary of ∂F in Ω:

$$d_F(x) := \operatorname{dist}(x, \Omega \cap \partial F).$$

The function

$$\tilde{d}_F(x) := \begin{cases} -d_F(x) & \text{if } x \in F, \\ d_F(x) & \text{if } x \in \Omega \setminus F, \end{cases}$$

is called the *signed distance function* from ∂F in Ω negative inside F. The distance from the empty set is assumed to be equal to $+\infty$. Observe that if $E \subset F$, then $\tilde{d}_E \geq \tilde{d}_F$ and for any $E, F \subseteq \Omega$, $F \neq \emptyset$,

$$\int_{E\Delta F} d_F\, dx = \int_{E\setminus F} \tilde{d}_F\, dx - \int_{F\setminus E} \tilde{d}_F\, dx = \int_E \tilde{d}_F\, dx - \int_F \tilde{d}_F\, dx,$$

provided $\displaystyle\int_{E\cap F} d_F dx < +\infty$. Moreover, we assume $\displaystyle\int_{E\Delta F} d_F dx := 0$ whenever $|E\Delta F| = 0$.

Given $\beta \in L^\infty(\partial\Omega)$, $E_0 \in BV(\Omega, \{0,1\})$ and $\lambda \geq 1$, recalling the definition of $\mathcal{C}_\beta(\cdot, \Omega)$ in (3.2), we define the *capillary Almgren-Taylor-Wang-type* functional $\mathcal{A}_\beta(\cdot, E_0, \lambda) : BV(\Omega, \{0,1\}) \to [-\infty, +\infty]$ with contact angle β, as

$$\mathcal{A}_\beta(E, E_0, \lambda) := \mathcal{C}_\beta(E, \Omega) + \lambda \int_{E\Delta E_0} d_{E_0}\, dx,$$

so that

$$\mathcal{A}_\beta(E, E_0, \lambda) = P(E, \Omega) + \lambda \int_E \tilde{d}_{E_0}\, dx - \int_{\partial\Omega} \beta\, \chi_E\, d\mathcal{H}^n - \lambda \int_{E_0} \tilde{d}_{E_0}\, dx \tag{3.29}$$

whenever $\displaystyle\int_{E\cap E_0} d_{E_0} dx < +\infty$.

3.3.1 Existence of minimizers of the functional $\mathcal{A}_\beta(\cdot, E_0, \lambda)$

We always suppose that $\lambda \geq 1$ and in this section we assume that

$$\begin{cases} E_0 \in BV(\Omega, \{0,1\}) \text{ is nonempty and bounded }, \\ \beta \in L^\infty(\partial\Omega) \text{ and } \exists \kappa \in (0, \frac{1}{2}] : -1 \leq \beta \leq 1 - 2\kappa\, \mathcal{H}^n\text{-a.e on } \partial\Omega. \end{cases} \tag{3.30}$$

Hence, there exists a cylinder $C_D^H = \hat{B}_D \times (0, H)$ containing E_0 whose basis is an open ball $\hat{B}_D \subset \mathbb{R}^n$ of radius $D > 0$ and height

$$H = 1 + \max\{x_{n+1} :\ x = (x', x_{n+1}) \in \overline{E_0}\}.$$

Define

$$R_0 := R_0(n, \kappa, E_0) = D + 1 + \max\Big\{8^{n^2+n+1} \left(\frac{P(E_0)}{\kappa}\right)^{\frac{n+1}{n}}, 4\mu(\kappa, n)\Big\}, \tag{3.31}$$

where $\mu(\kappa, n) = (1/\kappa + 2)^{\frac{n+1}{n}}$.

Theorem 3.12 (Existence of minimizers and uniform bound). *Suppose that* (3.30) *holds. Then the minimum problem*

$$\inf_{E\in BV(\Omega,\{0,1\})} \mathcal{A}_\beta(E,E_0,\lambda) \tag{3.32}$$

has a solution E_λ. *Moreover, any minimizer is contained in* $C^H_{R_0}$.

Proof. Let $f = \lambda \tilde{d}_{E_0}$ and

$$\mathcal{V} : BV(\Omega,\{0,1\}) \to (-\infty,+\infty], \quad \mathcal{V}(E) := \int_E f dx.$$

Then $\mathcal{V}$ satisfies Hypothesis 3.6 and by Remark 3.9 $\mathcal{R}_0 \le R_0$. Now the proof directly follows from Theorem 3.8. □

Remark 3.13. If $E_0 = \emptyset$, then (3.32) has a unique solution $E_\lambda = \emptyset$. Moreover, for some choices of $\lambda \ge 1$ and $\emptyset \ne E_0 \in BV(\Omega,\{0,1\})$, the empty set solves (3.32). For example, let B_ρ be the ball centered at x such that $x_{n+1} \ge 4\rho + 4$. If $\lambda\rho \le n$, then as in [36, 17], one can show that $E_\lambda = \emptyset$ is the unique minimizer of $\mathcal{A}_\beta(\cdot, B_\rho, \lambda)$.

Remark 3.14. Let F minimize $\mathcal{A}_\beta(\cdot,E_0,\lambda)$ in $BV(C^H_{R_0},\{0,1\})$. Then F is an unconstrained minimizer, i.e.

$$\mathcal{A}_\beta(F,E_0,\lambda) = \min_{E\in BV(\Omega,\{0,1\})} \mathcal{A}_\beta(E,E_0,\lambda). \tag{3.33}$$

Indeed, let E_λ be any minimizer of $\mathcal{A}_\beta(\cdot,E_0,\lambda)$. Clearly, $\mathcal{A}_\beta(F,E_0,\lambda) \ge \mathcal{A}_\beta(E_\lambda,E_0,\lambda)$. On the other hand, by Theorem 3.12 $E_\lambda \subseteq C^H_{R_0}$ and by minimality of F in $C^H_{R_0}$ we have $\mathcal{A}_\beta(F,E_0,\lambda) \le \mathcal{A}_\beta(E_\lambda,E_0,\lambda)$, which implies (3.33).

Recalling Remark 3.3 and definition (3.1) of $\mathcal{E}(E_0)$ we have also the following result.

Proposition 3.15 (Existence of constrained minimizers of $\mathcal{C}_\beta$). *Under assumptions* (3.30) *the constrained minimum problem*

$$\inf_{E\in BV(\Omega,\{0,1\}),\, E\in\mathcal{E}(E_0)} \mathcal{C}_\beta(E,\Omega) \tag{3.34}$$

has a solution. In addition, any minimizer E^+ *satisfies* $E^+ \subseteq C^H_{R_0}$, *where* R_0 *is given by* (3.31), *and* E^+ *is also a solution of*

$$\inf_{E\in BV(\Omega,\{0,1\}),\, E\in\mathcal{E}(E^+)} \mathcal{C}_\beta(E,\Omega).$$

Proof. Set

$$\mathcal{V} : BV(\Omega,\{0,1\}) \to [0,+\infty], \quad \mathcal{V}(E) := \begin{cases} 0 & \text{if } E \in \mathcal{E}(E_0), \\ +\infty & \text{if } E \in BV(\Omega,\{0,1\}) \setminus \mathcal{E}(E_0). \end{cases} \tag{3.35}$$

Then $\mathcal{V}$ satisfies Hypothesis 3.6 and $\mathcal{R}_0 \le R_0$. Now existence of a minimizer E^+ of $\mathcal{C}_\beta(\cdot,\Omega)$ in $\mathcal{E}(E_0)$ and the inclusion $E^+ \subseteq C^H_{R_0}$ follow from Theorem 3.8. To show the last statement we observe that the inclusion $E_0 \subseteq E^+$ implies $\mathcal{E}(E^+) \subseteq \mathcal{E}(E_0)$. Hence the minimality of E^+ yields the inequality $\mathcal{C}_\beta(E^+,\Omega) \le \mathcal{C}_\beta(E,\Omega)$ for any $E \in \mathcal{E}(E^+)$. □

Solutions of (3.34) will be called constrained minimizers of $\mathcal{C}_\beta(\cdot,\Omega)$ in $\mathcal{E}(E_0)$.

Example 3.16. Suppose that $E_0\subset\Omega$ is a closed convex set such that $\nu_{E_0}\cdot e_{n+1}\ge 0$ $\mathcal{H}^n$-a.e. on $\Omega\cap\partial E_0$. Then for every $\beta\in L^\infty(\partial\Omega,[-1,0])$ the set E_0 is a constrained minimizer of $\mathcal{C}_\beta(\cdot,\Omega)$ in $\mathcal{E}(E_0)$. Indeed, by Corollary 1.8 $P(E_0,\Omega)\le P(E,\Omega)$ for all $E\in\mathcal{E}(E_0)$, therefore

$$\mathcal{C}_\beta(E,\Omega)-\mathcal{C}_\beta(E_0,\Omega)=P(E,\Omega)-P(E_0,\Omega)+\int_{\partial\Omega}(-\beta)\chi_{E\setminus E_0}d\mathcal{H}^n\ge 0.$$

The following lemma shows the behaviour of E_λ as $\lambda\to+\infty$.

Lemma 3.17 (**Asymptotics of E_λ as time goes to 0^+**). *Assume (3.30) and $|\overline{E_0}\setminus E_0|=0$. Then any minimizer E_λ satisfies:*

a) $\lim\limits_{\lambda\to+\infty}|E_\lambda\Delta E_0|=0,$

b) $\lim\limits_{\lambda\to+\infty}\mathcal{C}_\beta(E_\lambda,\Omega)=\mathcal{C}_\beta(E_0,\Omega),$

c) $\lim\limits_{\lambda\to+\infty}\lambda\int_{E_\lambda\Delta E_0}d_{E_0}\,dx=0.$

d) if $\|\beta\|_\infty<1$, then $\overline{\Omega\cap\partial E_\lambda}\xrightarrow{K}\overline{\Omega\cap\partial E_0}$ as $\lambda\to+\infty$, where $\xrightarrow{K}$ denotes Kuratowski convergence [75].

Proof. a) We have

$$\kappa P(E_\lambda)\le\mathcal{A}_\beta(E_\lambda,E_0,\lambda)\le\mathcal{A}_\beta(E_0,E_0,\lambda)=\mathcal{C}_\beta(E_0,\Omega)\le P(E_0).$$

Moreover, from $\mathcal{A}_\beta(E_\lambda,E_0,\lambda)\le P(E_0)$ and (1.4) we get $\lambda\int_{E_\lambda\Delta E_0}d_{E_0}\,dx\le P(E_0)$, hence

$$\lim_{\lambda\to+\infty}\int_{E_\lambda\Delta E_0}d_{E_0}\,dx=0.\tag{3.36}$$

Recall from Theorem 3.12 that $E_\lambda\subseteq C^H_{R_0}$ for all $\lambda\ge 1$. Hence, by compactness, from every diverging sequence $\{\lambda_i\}$ we can select a subsequence $\{\lambda_{i_k}\}$ such that

$$E_{\lambda_{i_k}}\to E_\infty\quad\text{in } L^1(\Omega)$$

for some $E_\infty\in BV(C^H_{R_0},\{0,1\})$. From (3.36) we deduce that $\int_{E_\infty\Delta E_0}d_{E_0}\,dx=0$, and thus, since $d_{E_0}\ge 0$ and by assumption $|\overline{E_0}\setminus E_0|=0$, we get $|E_\infty\Delta E_0|=0$. Now arbitrariness of $\{\lambda_j\}$ implies a).

b) Clearly, $\mathcal{C}_\beta(E_\lambda,\Omega)\le\mathcal{A}_\beta(E_\lambda,E_0,\lambda)\le\mathcal{C}_\beta(E_0,\Omega)$ for all $\lambda\ge 1$. Then by a) and by the $L^1(\Omega)$-lower semicontinuity of $\mathcal{C}_\beta(\cdot,\Omega)$ (Lemma 3.4) we establish

$$\mathcal{C}_\beta(E_0,\Omega)\le\liminf_{\lambda\to+\infty}\mathcal{C}_\beta(E_\lambda,\Omega)\le\limsup_{\lambda\to+\infty}\mathcal{C}_\beta(E_\lambda,\Omega)\le\mathcal{C}_\beta(E_0,\Omega),$$

and b) follows.

c) follows from b) and nonnegativity of $\lambda \int_{E_\lambda \Delta E_0} d_{E_0}\, dx,$ since

$$\limsup_{\lambda\to+\infty} \lambda \int_{E_\lambda \Delta E_0} d_{E_0}\, dx \le \lim_{\lambda\to+\infty} [\mathcal{C}_\beta(E_0,\Omega) - \mathcal{C}_\beta(E_\lambda,\Omega)] = 0.$$

d) It suffices to show that every diverging sequence $\{\lambda_j\}$ has a subsequence $\{\lambda_j'\}$ such that

$$K - \lim_{j\to+\infty} \overline{\Omega\cap\partial E_{\lambda_j'}} = \overline{\Omega\cap\partial E_0}.$$

Choose any sequence $\lambda_j \to +\infty$. By compactness of closed sets in Kuratowski convergence [75, page 340], there exists a closed set $C \subset \overline{\Omega}$ such that up to a not relabelled subsequence $\overline{\Omega\cap\partial E_{\lambda_j}} \overset{K}{\to} C$ as $j\to+\infty$. Let us show first that $\overline{\Omega\cap\partial E_0} \subseteq C$. Take any $x\in\mathbb{R}^{n+1}\setminus C$; we may suppose that $x\in\Omega$. Since C is closed, there exists a ball $B_\rho(x)$ such that $B_\rho(x)\cap C=\emptyset$. Since $\overline{\Omega\cap\partial E_{\lambda_j}} \overset{K}{\to} C$ as $j\to+\infty$, we have $B_\rho(x)\cap\overline{\Omega\cap\partial E_{\lambda_j}}=\emptyset$ for $j\ge1$ large enough. Therefore, $P(E_{\lambda_j}, B_\rho(x)\cap\Omega)=0$, and by a) and lower semicontinuity, $P(E_0,B_\rho(x)\cap\Omega)=0$. This yields $B_{\rho/2}(x)\cap\overline{\Omega\cap\partial E_0}=\emptyset$ and thus $\mathbb{R}^{n+1}\setminus C\subseteq\mathbb{R}^{n+1}\setminus\overline{\Omega\cap\partial E_0}$.

Now suppose that there exists $x\in C\setminus\overline{\Omega\cap\partial E_0}$. Then there exists $\rho>0$ such that $B_\rho(x)\cap\overline{\Omega\cap\partial E_0}=\emptyset$. Since $x\in C$, there exists $x_j\in\overline{\Omega\cap\partial E_{\lambda_j}}$ such that $x_j\to x$. Choose $j\in\mathbb{N}$ so large that $x_j\in B_{\rho/4}(x)$ and $R(n,\kappa)\lambda_j^{-1/2}<\rho/4$, where $R(n,\kappa)$ is defined in (3.41). By Proposition 3.24 below, we have

$$d_{E_0}(x_j)\le R(n,\kappa)\lambda_j^{-1/2}<\frac{\rho}{4}.$$

On the other hand, by construction, $d_{E_0}(x)\ge\frac{3\rho}{4}$, which leads to a contradiction. This yields $C\subseteq\overline{\Omega\cap\partial E_0}$, and d) follows. □

Proposition 3.18 (Energy-dissipation balance). *Let* E_0,β *satisfy* (3.30), $\mathcal{M}_\lambda := \{E_\lambda\}\subset BV(\Omega,\{0,1\})$ *be the family of minimizers of* $\mathcal{A}_\beta(\cdot,E_0,\lambda)$, $\lambda\ge1$. *Then the maps*

$$\lambda\in[1,+\infty)\mapsto\int_{E_\lambda\Delta E_0} d_{E_0}dx,\quad \textit{and}\quad \lambda\in[1,+\infty)\mapsto\mathcal{C}_\beta(E_\lambda,\Omega)$$

are well-defined a.e. on $[1,+\infty)$*, i.e. independent of the choice of* $E_\lambda\in\mathcal{M}_\lambda$*. Moreover, for any* $1\le\mu\le\lambda$ *the following energy dissipation equation holds:*

$$\mathcal{A}_\beta(E_\lambda,E_0,\lambda)=\mathcal{A}_\beta(E_\mu,E_0,\mu)+\int_\mu^\lambda\Big(\int_{E_\eta\Delta E_0} d_{E_0}dx\Big)d\eta. \tag{3.37}$$

In particular,

$$\mathcal{C}_\beta(E_0,\Omega)=\mathcal{A}_\beta(E_\mu,E_0,\mu)+\int_\mu^{+\infty}\Big(\int_{E_\eta\Delta E_0} d_{E_0}dx\Big)d\eta. \tag{3.38}$$

Proof. Observe that the map $g:\lambda\in[1,+\infty)\mapsto\mathcal{A}_\beta(E_\lambda,E_0,\lambda)$ is well-defined, i.e. independent of choice $E_\lambda\in\mathcal{M}_\lambda$. Take any $1\le\mu<\lambda$, $E_\mu\in\mathcal{M}_\mu$, $E_\lambda\in\mathcal{M}_\lambda$. By minimality of E_λ and E_μ, we have

$$\begin{aligned} g(\lambda)-g(\mu)\le&\mathcal{A}_\beta(E_\mu,E_0,\lambda)-\mathcal{A}_\beta(E_\mu,E_0,\mu)=(\lambda-\mu)\int_{E_\mu\Delta E_0} d_{E_0}dx,\\ g(\lambda)-g(\mu)\ge&\mathcal{A}_\beta(E_\lambda,E_0,\lambda)-\mathcal{A}_\beta(E_\lambda,E_0,\mu)=(\lambda-\mu)\int_{E_\lambda\Delta E_0} d_{E_0}dx. \end{aligned} \tag{3.39}$$

From here it immediately follows that g is nondecreasing. By Theorem 3.12 $E_\eta \subset C^H_{R_0}$ for any $\eta \geq 1$ and, hence, g is Lipschitz. In particular, $g \in W^{1,\infty}$. Hence, from (3.39) for a.e. $\lambda \geq 1$ it follows that

$$\int_{E_\lambda \Delta E_0} d_{E_0} dx \leq g'(\lambda) \leq \int_{E_\lambda \Delta E_0} d_{E_0} dx.$$

Therefore, for a.e. $\lambda \geq 1$ the integral $\int_{E_\lambda \Delta E_0} d_{E_0} dx$ is independent of the choice of $E_\lambda \in \mathcal{M}_\lambda$ and equal to $g'(\lambda)$. Moreover, by the well-definedness of g, for such $\lambda \geq 1$ the value of $\mathcal{C}_\beta(E_\lambda, \Omega) = g(\lambda) - \int_{E_\lambda \Delta E_0} d_{E_0} dx$ is also independent of $E_\lambda \in \mathcal{M}_\lambda$. Now (3.37) follows directly from the fundamental theorem of calculus

$$g(\lambda) - g(\mu) = \int_\mu^\lambda g'(\eta) d\eta,$$

while (3.38) is a consequence of letting $\lambda \to +\infty$ in (3.37) and Lemma 3.17. □

Remark 3.19. If $1 \leq \mu < \lambda$ from (3.39) we deduce

$$\int_{E_\lambda \Delta E_0} dx \leq \int_{E_\mu \Delta E_0} dx, \quad \forall E_\lambda \in \mathcal{M}_\lambda,\ E_\mu \in \mathcal{M}_\mu$$

3.4 Density estimates and regularity of minimizers

In this section we assume that

$$\begin{cases} E_0 \in BV(\Omega, \{0,1\}) \text{ is nonempty and bounded }, \\ \beta \in L^\infty(\partial\Omega) \text{ and } \exists \kappa \in (0, \frac{1}{2}] : \ \|\beta\|_\infty \leq 1 - 2\kappa. \end{cases} \tag{3.40}$$

Define

$$R(n,\kappa) := \left(2^{n+3} \frac{\omega_n + (n+1)\omega_{n+1}}{\omega_{n+1}\kappa^{n+1}}\right)^{\frac{1}{2}}, \ \gamma(n,\kappa) := \frac{\kappa(n+1)}{\sqrt{R(n,\kappa)^2 + 4\kappa(n+1)} + R(n,\kappa)}, \tag{3.41}$$

and

$$C(n,\kappa) := (n+1)\omega_{n+1} + 2\omega_n + \frac{\kappa(n+1)}{2}\omega_{n+1}, \quad c(n,\kappa) := c_{n+1}\left(\frac{\kappa}{4}\right)^n,$$

where c_{n+1} is the relative isoperimetric constant for the ball. The aim of this section is to prove the following uniform density estimates for minimizers of $\mathcal{A}_\beta(\cdot, E_0, \lambda)$, needed to prove regularity of minimizers (Theorem 3.22) and Proposition 3.25.

Theorem 3.20. *Assume that E_0 and β are as in (3.40) and $E_\lambda \in BV(\Omega, \{0,1\})$ is a minimizer of $\mathcal{A}_\beta(\cdot, E_0, \lambda)$. Then either $E_\lambda = \emptyset$ or*

$$\left(\frac{\kappa}{4}\right)^{n+1} \leq \frac{|E_\lambda \cap B_r(x)|}{\omega_{n+1} r^{n+1}} \leq 1 - \left(\frac{\kappa}{4}\right)^{n+1}, \tag{3.42}$$

$$c(n,\kappa) \leq \frac{P(E_\lambda, B_r(x))}{r^n} \leq C(n,\kappa) \tag{3.43}$$

for every $x \in \partial E_\lambda$ and $r \in (0, \frac{\gamma(n,\kappa)}{\lambda^{1/2}})$. In particular,

$$\mathcal{H}^n(\partial E_\lambda \setminus \partial^* E_\lambda) = 0. \tag{3.44}$$

We postpone the proof after several auxiliary results. First we show a weaker version of Theorem 3.20; the difference stands in that Proposition 3.21 holds for $r \le O(\frac{1}{\lambda})$ and $O(\frac{1}{\lambda})$ depends on $E_0,$ whereas Theorem 3.20 is valid for $r \le O(\frac{1}{\lambda^{1/2}})$ and $O(\frac{1}{\lambda^{1/2}})$ is independent of $E_0.$

Proposition 3.21. *Under the assumptions of Theorem 3.20, setting*

$$\Lambda := \Lambda(\lambda, n, \kappa, P(E_0)) = \lambda\,\mathrm{diam}(\hat{B}_{D+R_0+1} \times (-1, H+1)),$$

for any nonempty $E_\lambda,$ $x \in \partial E_\lambda$ and $r \in (0, \min\{1, \frac{\kappa(n+1)}{2\Lambda}\}),$ the density estimates (3.42)-(3.43) hold.

Proof. For completeness we give the full proof of the proposition using the methods of [78, 91]. We recall that one could also employ the density estimates for almost minimizers of the capillary functional (see for instance [49, Lemma 2.8]).

Set $r_0 := \min\{1, \frac{\kappa(n+1)}{2\Lambda}\},$ and fix $x \in \partial^* E_\lambda.$ Let $B_r := B_r(x)$ be the ball of radius $r \in (0, r_0)$ centered at $x,$ we can choose r such that

$$\mathcal{H}^n(\partial B_r \cap \partial E_\lambda) = 0.$$

First we show that E_λ satisfies

$$\kappa P(E_\lambda \cap B_r) \le 2\mathcal{H}^n(E_\lambda \cap \partial B_r) + \Lambda |E_\lambda \cap B_r|. \tag{3.45}$$

Comparing $\mathcal{A}_\beta(E_\lambda, E_0, \lambda)$ with $\mathcal{A}_\beta(E_\lambda \setminus B_r, E_0, \lambda),$ for a.e. $s \in (r, r_0)$ we establish

$$\begin{aligned} P(E_\lambda, B_s \cap \Omega) - &\int_{B_r \cap \partial\Omega} \beta \chi_{E_\lambda \cap B_r}\, d\mathcal{H}^n + \lambda \int_{E_\lambda \cap B_r} \tilde{d}_{E_0} dy \\ &\le P(E_\lambda, (B_s \setminus \overline{B}_r) \cap \Omega) + \mathcal{H}^n(E_\lambda \cap \partial B_r). \end{aligned}$$

Sending $s \to r^+$ we get

$$P(E_\lambda, B_r \cap \Omega) - \int_{B_r \cap \partial\Omega} \beta \chi_{E_\lambda}\, d\mathcal{H}^n + \lambda \int_{E_\lambda \cap B_r} \tilde{d}_{E_0} dy \le \mathcal{H}^n(E_\lambda \cap \partial B_r). \tag{3.46}$$

By Theorem 3.12 $E_\lambda \subseteq C_{R_0}^H$ and thus, since $r_0 \le 1,$ for any $y \in B_r$

$$\lambda |\tilde{d}_{E_0}(y)| \le \lambda\,\mathrm{diam}(\hat{B}_{D+R_0+1} \times (-1, H+1)) = \Lambda. \tag{3.47}$$

Moreover, using (3.4) for $E_\lambda \cap B_r$ we get (3.45):

$$\begin{aligned} \kappa P(E_\lambda \cap B_r) \le& P(E_\lambda, B_r \cap \Omega) + \mathcal{H}^n(E_\lambda \cap \partial B_r) - \int_{B_r \cap \partial\Omega} \beta \chi_{E_\lambda}\, d\mathcal{H}^n \\ \le& 2\mathcal{H}^n(E_\lambda \cap \partial B_r) + \Lambda |E_\lambda \cap B_r|. \end{aligned}$$

Now by the isoperimetric inequality,

$$P(E_\lambda \cap B_r) \ge (n+1)\omega_{n+1}^{\frac{1}{n+1}} |E_\lambda \cap B_r|^{\frac{n}{n+1}}. \tag{3.48}$$

Set $m(r) := |E_\lambda \cap B_r|$. Then m is absolutely continuous, $m(0) = 0$, $m(r) > 0$ for all $r > 0$ and $m'(r) = \mathcal{H}^n(E_\lambda \cap \partial B_r)$ for a.e. $r \in (0, r_0)$. Consequently, (3.45) and (3.48) give

$$\kappa(n+1)\omega_{n+1}^{\frac{1}{n+1}} m(r)^{\frac{n}{n+1}} \le 2m'(r) + \Lambda m(r) = 2m'(r) + \Lambda m(r)^{\frac{n}{n+1}} m(r)^{\frac{1}{n+1}}. \tag{3.49}$$

Since $m(r) \le \omega_{n+1} r^{n+1}$ and $r \le \frac{\kappa(n+1)}{2\Lambda}$, from the last inequality we obtain

$$\frac{\kappa}{4}(n+1)\omega_{n+1}^{\frac{1}{n+1}} m(r)^{\frac{n}{n+1}} \le m'(r).$$

Integrating we get the lower volume density estimate

$$m(r) \ge \left(\frac{\kappa}{4}\right)^{n+1} \omega_{n+1} r^{n+1}, \quad \forall r \in (0, r_0).$$

Let us prove the upper volume density estimate in (3.42). Since $E_\lambda \subseteq \Omega$ if $x \in \partial\Omega \cap \partial^* E_\lambda$, the inequality

$$\frac{|B_r \setminus E_\lambda|}{\omega_{n+1} r^{n+1}} \ge \frac{1}{2} > \left(\frac{\kappa}{4}\right)^{n+1} \qquad \forall r > 0 \tag{3.50}$$

is trivial. So assume that $x \in \Omega \cap \partial^* E_\lambda$. Since $\mathcal{A}_\beta(E_\lambda, E_0, \lambda) \le \mathcal{A}_\beta((E_\lambda \cup B_r) \cap \Omega, E_0, \lambda)$, arguing as in the proof of (3.46) we get

$$P(E_\lambda, B_r \cap \Omega) + \int_{\partial\Omega} \beta \chi_{(B_r\cap\Omega)\setminus E_\lambda}\, d\mathcal{H}^n \le \mathcal{H}^n((\Omega \setminus E_\lambda) \cap \partial B_r) + \lambda \int_{(B_r\cap\Omega)\setminus E_\lambda} \tilde{d}_{E_0} dy. \tag{3.51}$$

From the isoperimetric inequality, (3.4), (3.51) and also (3.47), it follows that

$$\begin{aligned} \kappa(n+1)\omega_{n+1}^{\frac{1}{n+1}} |(B_r \setminus E_\lambda) \cap \Omega|^{\frac{n}{n+1}} &\le \kappa P((B_r \setminus E_\lambda) \cap \Omega) \le \mathcal{C}_{-\beta}((B_r \setminus E_\lambda) \cap \Omega, \Omega) \\ &\le P(E_\lambda, B_r \cap \Omega) + \int_{\partial\Omega} \beta \chi_{(B_r\cap\Omega)\setminus E_\lambda}\, d\mathcal{H}^n + \mathcal{H}^n((\Omega \setminus E_\lambda) \cap \partial B_r) \\ &\le 2\mathcal{H}^n((\Omega \setminus E_\lambda) \cap \partial B_r) + \Lambda |(B_r \setminus E_\lambda) \cap \Omega|. \end{aligned} \tag{3.52}$$

Repeating the same arguments as before we establish

$$\frac{|B_r \setminus E_\lambda|}{\omega_{n+1} r^{n+1}} \ge \frac{|(B_r \setminus E_\lambda) \cap \Omega|}{\omega_{n+1} r^{n+1}} \ge \left(\frac{\kappa}{4}\right)^{n+1} \qquad \forall r \in (0, r_0).$$

Let us now show (3.43). From (3.46) we get

$$\begin{aligned} P(E_\lambda, B_r) =& P(E_\lambda, B_r \cap \Omega) + \int_{B_r \cap \partial\Omega} \chi_{E_\lambda}\, d\mathcal{H}^n \\ &\le \mathcal{H}^n(E_\lambda \cap \partial B_r) + \int_{B_r\cap\partial\Omega} (1+\beta)\chi_{E_\lambda}\, d\mathcal{H}^n + \Lambda |E_\lambda \cap B_r| \\ &\le (n+1)\omega_{n+1} r^n + 2\omega_n r^n + \omega_{n+1} r^n (\Lambda r) \\ &\le \left[(n+1)\omega_{n+1} + 2\omega_n + \omega_{n+1}\frac{\kappa(n+1)}{2}\right] r^n \end{aligned}$$

for a.e $r \in (0, r_0)$. Since $P(E_\lambda, \cdot)$ is a nonnegative measure, this inequality holds for all $r \in (0, r_0)$. This proves the upper perimeter estimate in (3.43).

The lower perimeter density estimate in (3.43) follows from (3.42) and the relative isoperimetric inequality (see for example [11, page 152]). □

Theorem 3.22 (Regularity of minimizers up to the boundary). *Assume that E_0 and β satisfy (3.40). Then any nonempty minimizer E_λ is open in $\mathbb{R}^{n+1}$ and $\Omega \cap \partial^* E_\lambda$ is an n-dimensional manifold of class $C^{2,\alpha}$ for a suitable $\alpha \in (0,1)$, and $\mathcal{H}^s((\partial E_\lambda \setminus \partial^* E_\lambda) \cap \Omega) = 0$ for all $s > n-7$. Moreover, if $\beta \in \mathrm{Lip}(\partial\Omega)$, then*

a) $\mathcal{H}^n((\partial E_\lambda \cap \partial\Omega) \Delta (\mathrm{Tr}(E_\lambda))) = 0$;

b) $\partial E_\lambda \cap \partial\Omega$ is a set of finite perimeter in $\partial\Omega$ and

$$\mathcal{H}^{n-1}(\partial(\partial E_\lambda \cap \partial\Omega) \setminus \partial^*(\partial E_\lambda \cap \partial\Omega)) = 0,$$

where $\partial(\partial E_\lambda \cap \partial\Omega)$ denotes the boundary of $\partial E_\lambda \cap \partial\Omega$ in $\partial\Omega$. Moreover, if $M_\lambda = \overline{\Omega \cap \partial E_\lambda}$, then

$$\partial(\partial E_\lambda \cap \partial\Omega) = M_\lambda \cap \partial\Omega.$$

c) There exists a relatively closed set $\Sigma \subset M_\lambda$ with $\mathcal{H}^{n-1}(\Sigma \cap \partial\Omega) = 0$ such that in a neighborhood of any $x \in (M_\lambda \cap \partial\Omega) \setminus \Sigma$ the set M_λ is a $C^{1,1/2}$-manifold with boundary, and

$$\nu_{E_\lambda} \cdot e_{n+1} = \beta \quad \text{on } (M_\lambda \cap \partial\Omega) \setminus \Sigma.$$

Proof. Since E_λ is a minimizer of $\mathcal{A}_\beta(\cdot, E_0, \lambda)$ in every ball $B \subset \Omega$, we can apply [82, Theorem 5.2] to prove that E_λ is open and $\Omega \cap \partial^* E_\lambda$ is $C^{2,\alpha}$ with $\mathcal{H}^s((\partial E_\lambda \setminus \partial^* E_\lambda) \cap \Omega) = 0$ for all $s > n-7$. Moreover, if $\beta \in \mathrm{Lip}(\partial\Omega)$, by (3.47) the remaning assertions follow from [49, Lemma 2.16, Theorem 1.10]. □

Remark 3.23. (Compare with [78, Remark 1.4] and [91].)

a) Assume that $x \in \overline{E_\lambda}$ and $r > 0$ are such that $B_r(x) \cap E_0 = \emptyset$. Then $d_{E_0} \geq 0$ in $E_\lambda \cap B_r(x)$ and from (3.46) we get

$$P(E_\lambda, B_r \cap \Omega) - \int_{B_r \cap \partial\Omega} \beta \chi_{E_\lambda} \, d\mathcal{H}^n \leq \mathcal{H}^n(E_\lambda \cap \partial B_r). \tag{3.53}$$

Then proceeding as in the proof of Proposition 3.21 we get $|E_\lambda \cap B_r| \geq (\kappa/2)^{n+1} \omega_{n+1} r^{n+1}$. Moreover, from (3.53) it follows that

$$P(E_\lambda, B_r \cap \Omega) \leq \mathcal{H}^n(E_\lambda \cap \partial B_r) + \int_{B_r \cap \partial\Omega} \chi_{E_\lambda} \, d\mathcal{H}^n \leq \big[(n+1)\omega_{n+1} + \omega_n\big] r^n.$$

b) Similarly, if $x \in \overline{E_\lambda}$ and $B_r(x) \cap (\Omega \setminus E_0) = \emptyset$, then $|B_r \setminus E_\lambda| \geq (\kappa/2)^{n+1} \omega_{n+1} r^{n+1}$.

Observe that in both cases r *need not be* in $(0, \min\{1, \frac{\kappa(n+1)}{2\Lambda}\})$ and the assumption $x \in \partial E_\lambda$ is not necessary.

The following proposition is the analog of [78, Lemma 2.1] and [91, Proposition 3.2.1].

Proposition 3.24 (L^∞ **-bound for the distance function).** *Assume that* E_0 *and* β *are as in* (3.40) *and* $E_\lambda \in BV(\Omega, \{0,1\})$ *is a minimizer of* $\mathcal{A}_\beta(\cdot, E_0, \lambda)$. *Then*

$$\sqrt{\lambda}\|d_{E_0}\|_{L^\infty(E_\lambda \Delta E_0)} \le R(n,\kappa). \tag{3.54}$$

Proof. Let $R := R(n,\kappa)$. Suppose by contradiction that there exist $\varepsilon > 0$, $\lambda \ge 1$ and $x \in E_\lambda \Delta E_0$ such that $d_{E_0}(x) > (R+\varepsilon)\lambda^{-1/2}$. Consider first the case $x \in E_\lambda \setminus E_0$. By regularity of E_λ (Theorem 3.22) we may assume that $x \in \partial E_\lambda \setminus E_0$. Note that $B_\rho \cap E_0 = \emptyset$, where $B_\rho := B_\rho(x)$, $\rho = (R+\varepsilon)\lambda^{-1/2}/2$. Since $\mathcal{A}_\beta(E_\lambda, E_0, \lambda) \le \mathcal{A}_\beta(E_\lambda \setminus B_\rho, E_0, \lambda)$, and $\tilde{d}_{E_0}(y) = d_{E_0}(y) \ge \rho$ for any $y \in B_\rho \cap E_\lambda$, from (3.46) we establish

$$\frac{(R+\varepsilon)\lambda^{1/2}}{2}|E_\lambda \cap B_\rho| \le \lambda \int_{E_\lambda \cap B_\rho} \tilde{d}_{E_0} dy \le \mathcal{H}^n(E_\lambda \cap \partial B_\rho) + \int_{B_\rho \cap \partial\Omega} \beta\chi_{E_\lambda}\, d\mathcal{H}^n \le [\omega_{n+1}(n+1) + \omega_n]\rho^n.$$

This and Remark 3.23 (a) yield[2]

$$\omega_{n+1}\frac{(R+\varepsilon)\kappa^{n+1}}{2^{n+2}}\lambda^{1/2}\rho^{n+1} \le [\omega_{n+1}(n+1)+\omega_n]\rho^n,$$

or equivalently, recalling the definition of ρ

$$(R+\varepsilon)^2 \le 2^{n+3}\frac{\omega_n + (n+1)\omega_{n+1}}{\omega_{n+1}\kappa^{n+1}} = R^2,$$

which is a contradiction. A similar contradiction is obtained when $x \in E_0 \setminus E_\lambda$. $\square$

Proof of Theorem 3.20. We repeat the same procedures of the proof of Proposition 3.21 with improved estimates for the volume term of $\mathcal{A}_\beta(\cdot, E_0, \lambda)$. Let $R := R(n,\kappa)$, $\gamma := \gamma(n,\kappa)$. Fix $x \in \partial^* E_\lambda$, and choose $r \in (0, \gamma\lambda^{-1/2})$ such that $\mathcal{H}^n(\partial B_r \cap \partial E_\lambda) = 0$. From (3.54) it follows

$$\sup_{(E_\lambda \setminus E_0)\cap B_r} d_{E_0} \le R\lambda^{-1/2}.$$

Therefore, using the obvious inequality

$$\sup_{(E_\lambda \cap E_0)\cap B_r} d_{E_0} \le 2r + \sup_{(E_0 \setminus E_\lambda)\cap B_r} d_{E_0} \le (2\gamma + R)\lambda^{-1/2},$$

from (3.46) we establish that

$$P(E_\lambda, B_r \cap \Omega) - \int_{B_r \cap \partial\Omega} \beta\chi_{E_\lambda}\, d\mathcal{H}^n + \le \mathcal{H}^n(E_\lambda \cap \partial B_r) + (R+2\gamma)\lambda^{1/2}|E_\lambda \cap B_r|. \tag{3.55}$$

Since $m(r) := |E_\lambda \cap B_r| \le \omega_{n+1}r^{n+1}$ and $r \le \frac{\gamma}{\lambda^{1/2}}$, similarly to (3.49) from (3.55) we deduce

$$\kappa(n+1)\omega_{n+1}^{\frac{1}{n+1}} m(r)^{\frac{n}{n+1}} \le 2m'(r) + (R+2\gamma)\lambda^{1/2} r\omega_{n+1}^{\frac{1}{n+1}} m(r)^{\frac{n}{n+1}}, \text{ for a.e. } r \in (0, \gamma\lambda^{1/2}).$$

[2] Since the upper bound for the radii in Proposition 3.21 is of order $O(\frac{1}{\lambda})$, in general, we cannot apply it with ρ.

By the definition of γ one has

$$(R+2\gamma)\lambda^{1/2}r \le (R+2\gamma)\gamma = \frac{1}{2}\,\kappa(n+1).$$

Thus,

$$\frac{\kappa}{4}\,(n+1)\omega_{n+1}^{\frac{1}{n+1}} m(r)^{\frac{n}{n+1}} \le m'(r) \quad \text{for a.e. } r\in(0,\gamma\lambda^{-1/2}).$$

Integrating this differential inequality we get the lower volume density estimate in (3.42).

Let us prove the upper volume density estimate in (3.42). Due to (3.50) we may suppose that $x\in\Omega\cap\partial^*E_\lambda$. As above one can estimate d_{E_0} in $(B_r\setminus E_\lambda)\cap\Omega$ as follows:

$$\sup_{\Omega\cap((B_r\setminus E_\lambda)\setminus E_0)} d_{E_0} \le 2r + \sup_{E_\lambda\Delta E_0} d_{E_0} \le (2\gamma+R)\lambda^{-1/2}. \tag{3.56}$$

Since $\tilde d_{E_0}\le 0$ in $\Omega\cap((B_r\setminus E_\lambda)\cap E_0)$, plugging (3.56) in (3.51) and proceeding as above we establish

$$\frac{\kappa}{4}\,(n+1)\omega_{n+1}^{\frac{1}{n+1}}|(B_r\setminus E_\lambda)\cap\Omega|^{\frac{n}{n+1}} \le \mathcal{H}^n((\Omega\setminus E_\lambda)\cap B_r),$$

from which the upper volume density estimates in (3.42) follows.

The proof of (3.43) is exactly the same as the proof of perimeter density estimates in Proposition 3.21. Finally, (3.44) is a standard consequence of a covering argument. □

Let us prove the following L^1-estimate for the minimizers of $\mathcal{A}_\beta(\cdot,E_0,\lambda)$, the analog of [78, Lemma 1.5] and [91, Proposition 3.2.3]. Notice carefully the exponent $-1/2$ of λ in (3.57).

Proposition 3.25 (L^1**-estimate).** *Assume that E_0 and β satisfy* (3.40) *and the uniform volume density estimates* (3.42) *holds for E_0. Then for any minimizer E_λ of $\mathcal{A}_\beta(\cdot,E_0,\lambda)$ the estimate*

$$|E_\lambda\Delta E_0| \le C_{n,\kappa}P(E_0)\,\ell + \frac{1}{\ell}\int_{E_\lambda\Delta E_0} d_{E_0}dx, \quad \ell\in\left(0,\frac{\gamma(n,\kappa)}{\lambda^{1/2}}\right) \tag{3.57}$$

holds, where

$$C_{n,\kappa} := \left(\frac{8}{\kappa}\right)^{n+1}\omega_{n+1}^{\frac{1}{n+1}}\,\mathfrak{b}(n)\,c_{n+1} \tag{3.58}$$

and $\mathfrak{b}(n)$ is the constant in Besicovitch covering theorem.

Proof. Set

$$A:=\{x\in E_\lambda\Delta E_0:\, d_{E_0}(x)\ge\ell\},\quad B:=\{x\in E_\lambda\Delta E_0:\, d_{E_0}(x)<\ell\}.$$

By Chebyshev inequality

$$|A|\le\frac{1}{\ell}\int_{E_\lambda\Delta E_0} d_{E_0}dx.$$

Let us estimate $|B|$. Since E_0 is bounded, by Besicovitch's covering theorem there exist at most countably many balls $\{B_\ell(x_i)\}$, $x_i\in\partial E_0$ such that any point of ∂E_0 belongs to at most $\mathfrak{b}(n)$

balls, $\partial E_0 \subset \bigcup_i B_\ell(x_i)$ and $B \subset \bigcup_i B_{2\ell}(x_i)$. Since the balls $\{B_{2\ell}(x_i)\}$ cover B, by the density estimates (3.42) and the relative isoperimetric inequality we get

$$\begin{aligned}|B_{2\ell}(x_i)| =& 2^{n+1}\omega_{n+1}\ell^{n+1} \le 2^{n+1}\left(\frac{4}{\kappa}\right)^{n+1} \min\{|B_\ell(x_i)\cap E_0|, |B_\ell(x_i)\setminus E_0|\}\\ &\le \left(\frac{8}{\kappa}\right)^{n+1} \omega_{n+1}^{\frac{1}{n+1}}\,\ell\,\min\{|B_\ell(x_i)\cap E_0|, |B_\ell(x_i)\setminus E_0|\}^{\frac{n}{n+1}}\\ &\le \left(\frac{8}{\kappa}\right)^{n+1} \omega_{n+1}^{\frac{1}{n+1}}\,\ell\, c_{n+1}\, P(E_0, B_\ell(x_i)).\end{aligned}$$

Therefore

$$|B| \le \left(\frac{8}{\kappa}\right)^{n+1} \omega_{n+1}^{\frac{1}{n+1}}\, c_{n+1}\,\ell \sum_i P(E_0, B_\ell(x_i)) \le \left(\frac{8}{\kappa}\right)^{n+1} \omega_{n+1}^{\frac{1}{n+1}}\, \mathfrak{b}(n)\, c_{n+1}\, P(E_0)\,\ell.$$

Now (3.57) follows from the estimates for $|A|, |B|$ and from $|E_\lambda \Delta E_0| \le |A| + |B|$. □

A specific choice of ℓ will be made in the proof of Theorem 3.38. We conclude this section with a proposition about the regularity of minimizers of $\mathcal{C}_\beta(\cdot,\Omega)$.

Proposition 3.26 (Density estimates for constrained minimizers of $\mathcal{C}_\beta$). *Assume that E_0 and β satisfy (3.40) and there exist $c_1, c_2, \varepsilon \in (0,1)$ such that for every $x \in \partial E_0$ and $r \in (0,\varepsilon)$ the inequalities*

$$c_1 \le \frac{|B_r(x)\cap E_0|}{|B_r(x)|} \le c_2$$

hold. Let E^+ be a constrained minimizer of $\mathcal{C}_\beta(\cdot,\Omega)$ in $\mathcal{E}(E_0)$. Then for every $x \in \partial E^+$ and $r \in (0,\varepsilon)$

$$\begin{aligned} c_1\left(\frac{\kappa}{8}\right)^{n+1} \le \frac{|B_r(x)\cap E^+|}{|B_r(x)|} \le 1 - \left(\frac{\kappa}{4}\right)^{n+1},\\ c_{n+1}c_1^{\frac{n}{n+1}}(\kappa/8)^n \le \frac{P(E^+, B(x,r))}{r^n} \le (n+1)\omega_{n+1} + \omega_n. \end{aligned} \tag{3.59}$$

In particular, $\mathcal{H}^n(\partial E^+ \setminus \partial^* E^+) = 0$.

Proof. Let $x \in \partial E^+$, and $r \in (0,\varepsilon)$ be such that $\mathcal{H}^n(\partial B_r \cap \partial^* E^+) = 0$, where $B_r := B_r(x)$.

We start with the upper volume density estimate in (3.59). We may suppose $x \in \Omega \cap \partial^* E^+$, since the case $x \in \partial\Omega \cap \partial^* E^+$ is trivial. Using $\mathcal{C}_\beta(E^+,\Omega) \le \mathcal{C}_\beta((E^+ \cup B_r)\cap\Omega, \Omega)$, as in (3.51) we establish

$$P(E^+, B_r) + \int_{\partial\Omega} \beta\chi_{(B_r\setminus E^+)\cap\Omega}\, d\mathcal{H}^n \le \mathcal{H}^n((\Omega\setminus E^+)\cap\partial B_r). \tag{3.60}$$

Adding $\mathcal{H}^n(\partial B_r \cap (\Omega \setminus E^+))$ to both sides and proceeding as in (3.52) we get

$$\kappa(n+1)\omega_{n+1}^{\frac{1}{n+1}}|(B_r\setminus E^+)\cap\Omega|^{\frac{n}{n+1}} \le 2\mathcal{H}^n((\Omega\setminus E^+)\cap\partial B_r)$$

and hence as in the proof of Theorem 3.20

$$|B_r \setminus E^+| \geq \left(\frac{\kappa}{4}\right)^{n+1} \omega_{n+1} r^{n+1}.$$

This implies the upper volume density estimate in (3.59).

The lower volume density estimate is a little delicate, since in general we cannot use the set $E = E^+ \setminus B_r$ as a competitor since it need not belong to $\mathcal{E}(E_0)$. If $d := d_{E_0}(x) = 0$, then $x \in \partial E_0$ and, hence, using $E_0 \cap B_r \subset E^+ \cap B_r$ and the lower volume density estimate for E_0 we establish

$$\frac{|E^+ \cap B_r|}{|B_r|} \geq \frac{|E_0 \cap B_r|}{|B_r|} \geq c_1 \geq c_1 \left(\frac{\kappa}{8}\right)^{n+1}.$$

If $d > 0$ and $r \in (0, \min\{\varepsilon, d\})$, then we may use comparison set $E^+ \setminus B_r$ and as in the proof of (3.42) we obtain

$$\frac{|E^+ \cap B_r|}{|B_r|} \geq \left(\frac{\kappa}{4}\right)^{n+1} \geq c_1 \left(\frac{\kappa}{8}\right)^{n+1}. \tag{3.61}$$

Suppose $d < \varepsilon$. Since one can extend (3.61) to $(0, d]$ by continuity, if $r \in (d, \min\{2d, \varepsilon\})$, then

$$\frac{|E^+ \cap B_r|}{|B_r|} \geq \frac{|E^+ \cap B_d|}{|B_d|} \cdot \left(\frac{d}{r}\right)^{n+1} \geq \left(\frac{\kappa}{8}\right)^{n+1} \geq c_1 \left(\frac{\kappa}{8}\right)^{n+1}.$$

Let $r \in [2d, \varepsilon)$ and $x_0 \in \overline{\Omega \cap \partial E_0}$ be such that $d = |x - x_0|$. Then using $B(x, r) \supset B(x_0, r - d)$, the lower density estimate for E_0 and $r - d \geq r/2$, we obtain

$$\frac{|E^+ \cap B_r|}{|B_r|} \geq \frac{|E_0 \cap B_{r-d}(x_0)|}{|B_{r-d}(x_0)|} \cdot \left(\frac{r-d}{r}\right)^{n+1} \geq c_1 \left(\frac{1}{2}\right)^{n+1} \geq c_1 \left(\frac{\kappa}{8}\right)^{n+1}.$$

Now the lower perimeter estimate follows from the volume density estimates and the relative isoperimetric inequality. The upper perimeter estimate is obtained from (3.60):

$$P(E^+, B_r) \leq \mathcal{H}^n((\Omega \setminus E^+) \cap \partial B_r) - \int_{\partial\Omega} \beta \chi_{(B_r \setminus E^+) \cap \Omega} \, d\mathcal{H}^n \leq ((n+1)\omega_{n+1} + \omega_n) r^n.$$

Finally, the relation $\mathcal{H}^n(\partial E^+ \setminus \partial^* E^+) = 0$ is a consequence of the density estimates together with a covering argument. □

3.5 Comparison principles

The main result of this section is the following comparison between minimizers of $\mathcal{A}_\beta(\cdot, E_0, \lambda)$.

Theorem 3.27 (Comparison for minimizers of $\mathcal{A}_\beta$). *Assume that $E_0, F_0,$ β_1, β_2 satisfy (3.30). Suppose that $E_0 \subseteq F_0$ and $\beta_1 \leq \beta_2$. Then*

a) there exists a minimizer F_λ^* of $\mathcal{A}_{\beta_2}(\cdot, F_0, \lambda)$ containing any minimizer of $\mathcal{A}_{\beta_1}(\cdot, E_0, \lambda)$;

b) there exists a minimizer $E_{\lambda *}$ of $\mathcal{A}_{\beta_1}(\cdot, E_0, \lambda)$ contained in any minimizer of $\mathcal{A}_{\beta_2}(\cdot, F_0, \lambda)$.

If in addition

$$\operatorname{dist}(\Omega \cap \partial E_0, \Omega \cap \partial F_0) > 0, \tag{3.62}$$

then all minimizers E_λ *and* F_λ *of* $\mathcal{A}_{\beta_1}(\cdot, E_0, \lambda)$ *and* $\mathcal{A}_{\beta_2}(\cdot, F_0, \lambda)$ *respectively satisfy*

$$E_\lambda \subseteq F_\lambda.$$

Remark 3.28. We do not exclude the case that either E_λ or F_λ is empty.

Remark 3.29. For any E_0, β satisfying (3.30), using Theorem 3.27 with $\beta_1 = \beta_2 = \beta$ and $F_0 = E_0$, we establish the existence of unique minimizers $E_{\lambda *}$ and E_λ^* of $\mathcal{A}_\beta(\cdot, E_0, \lambda)$ such that any other minimizer E_λ satisfies $E_{\lambda *} \subseteq E_\lambda \subseteq E_\lambda^*$.

Definition 3.30 (Maximal and minimal minimizers). *We call* E_λ^* *and* $E_{\lambda *}$ *the maximal and minimal minimizer of* $\mathcal{A}_\beta(\cdot, E_0, \lambda)$ *respectively.*

Before proving Theorem 3.27 we need the following observations. Given β satisfying (3.30), $C = C_r^h$, $h, r > 0$ and $v \in L^\infty_{\mathrm{loc}}(\Omega)$, $v \geq 0$ a.e. in $\Omega \setminus C$, define the convex functional $\mathcal{B}_\beta(\cdot, v, C) : BV(\Omega, [0,1]) \to (-\infty, +\infty]$, a sort of level-set capillary Almgren-Taylor-Wang-type functional, as

$$\mathcal{B}_\beta(u, v, C) = \mathtt{C}_\beta(u, \Omega) + \int_\Omega uv\, dx.$$

Set

$$\mathcal{R}_1(C, v) := r + 1 + \max\left\{8^{n^2+n+1}\left(\frac{\mathcal{C}_\beta(C,\Omega) + \|v\|_{L^\infty(C)}|C|}{\kappa}\right)^{\frac{n+1}{n}}, 4\mu(\kappa, n)\right\},$$

where $\mu(\kappa, n) = (1/\kappa + 2)^{\frac{n+1}{n}}$. By Example 3.7 the functional

$$\mathcal{V} : BV(\Omega, \{0,1\}) \to (-\infty, +\infty], \qquad \mathcal{V}(E) := \int_E v dx$$

satisfies Hypothesis 3.6. Thus, by Theorem 3.8 the functional $E \in BV(\Omega, \{0,1\}) \mapsto \mathcal{B}_\beta(\chi_E, v, C) \in \mathbb{R}$ has a minimizer, and every minimizer E_v satisfies

$$E_v \subseteq C^h_{\mathcal{R}_1(C,v)}. \tag{3.63}$$

Notice that by (1.10) and (3.3),

$$\mathcal{B}_\beta(u, v, C) = \int_0^1 \mathcal{B}_\beta(\chi_{\{u>t\}}, v, C)\, dt \quad \forall u \in BV(\Omega, [0,1]), \tag{3.64}$$

which yields that χ_{E_v} is a minimizer of $\mathcal{B}_\beta(\cdot, v, C)$ in $BV(\Omega, [0,1])$.

The following remark is in the spirit of [24, Section 1].

Remark 3.31 (Minimality of level sets). From (3.64) it follows that $u \in BV(\Omega, [0,1])$ is a minimizer of $\mathcal{B}_\beta(\cdot, v, C)$ in $BV(\Omega, [0,1])$ if and only if $\chi_{\{u>t\}}$ is a minimizer of $\mathcal{B}_\beta(\cdot, v, C)$ for a.e. $t \in [0,1]$. Indeed, let for some $u \in BV(\Omega, [0,1])$ the function $\chi_{\{u>t\}}$ be a minimizer of

$\mathcal{B}_\beta(\cdot, v, C)$ for a.e. $t \in [0,1]$. Then for any $w \in BV(\Omega,[0,1])$ and for a.e. $t \in [0,1]$ one has $\mathcal{B}_\beta(w,v,C) \ge \mathcal{B}_\beta(\chi_{\{u>t\}}, v, C)$, therefore,

$$\mathcal{B}_\beta(u,v,C) = \int_0^1 \mathcal{B}_\beta(\chi_{\{u>t\}}, v, C)dt \le \mathcal{B}_\beta(w,v,C).$$

Conversely, if $u \in BV(\Omega,[0,1])$ is a minimizer of $\mathcal{B}_\beta(\cdot,v,C)$, then for a.e. $t \in [0,1]$ one has $\mathcal{B}_\beta(u,v,C) \le \mathcal{B}_\beta(\chi_{\{u>t\}}, v, C)$. Hence, from (3.64) it follows that $\mathcal{B}_\beta(u,v,C) = \mathcal{B}_\beta(\chi_{\{u>t\}}, v, C)$ for a.e. $t \in [0,1]$. In particular, if $u \in BV(\Omega,[0,1])$ is a minimizer of $\mathcal{B}_\beta(\cdot,v,C)$, then by (3.63) $\{u > t\} \subseteq C^h_{\mathcal{R}_1(C,v)}$ for a.e. $t \in [0,1]$, i.e. $u = 0$ a.e. in $\Omega \setminus C^h_{\mathcal{R}_1(C,v)}$. Hence,

$$\min_{u \in BV(\Omega,[0,1])} \mathcal{B}_\beta(u,v,C) = \min_{u \in BV(\Omega,[0,1]),\, u=0 \text{ a.e. in } \Omega\setminus C^h_{\mathcal{R}_1(C,v)}} \mathcal{B}_\beta(u,v,C). \tag{3.65}$$

Lemma 3.32. *Let E_0, β satisfy (3.30), and R_0 be defined as in (3.31). Then E_λ is a minimizer of $\mathcal{A}_\beta(\cdot,E_0,\lambda)$ if and only if χ_{E_λ} is a minimizer of $\mathcal{B}_\beta(\cdot, v^\lambda_{E_0}, C^H_{R_0})$, where $v^\lambda_{E_0} = \lambda \chi_{C^H_{R_0}} \tilde{d}_{E_0}$.*

Proof. By (3.29) we have

$$\mathcal{A}_\beta(E,E_0,\lambda) = \mathcal{B}_\beta(\chi_E, v^\lambda_{E_0}, C^H_{R_0}) - \lambda \int_{E_0} \tilde{d}_{E_0}\, dx \qquad \forall E \in BV(C^H_{R_0}, \{0,1\}). \tag{3.66}$$

Now if E_λ minimizes $\mathcal{A}_\beta(\cdot,E_0,\lambda)$, we have $E_\lambda \subseteq C^H_{R_0}$ (Theorem 3.12) and thus, for any $u \in BV(\Omega,[0,1])$ with $u=0$ a.e. in $\Omega \setminus C^H_{R_0}$ from (3.64)-(3.66) we deduce

$$\begin{aligned}\mathcal{B}_\beta(u, v^\lambda_{E_0}, C^H_{R_0}) &= \int_0^1 \mathcal{B}_\beta(\chi_{\{u>t\}}, v^\lambda_{E_0}, C^H_{R_0})\, dt = \int_0^1 \mathcal{A}_\beta(\{u>t\}, E_0, \lambda)\, dt + \lambda \int_{E_0} \tilde{d}_{E_0}\, dx \\ &\ge \int_0^1 \mathcal{A}_\beta(E_\lambda, E_0, \lambda)\, dt + \lambda \int_{E_0} \tilde{d}_{E_0}\, dx = \mathcal{B}_\beta(\chi_{E_\lambda}, v^\lambda_{E_0}, C^H_{R_0}).\end{aligned}$$

By (3.65) χ_{E_λ} is a minimizer of $\mathcal{B}_\beta(\cdot, v^\lambda_{E_0}, C^H_{R_0})$.

Conversely, assume that χ_{E_λ} is a minimizer of $\mathcal{B}_\beta(\cdot, v^\lambda_{E_0}, C^H_{R_0})$, then by (3.66) $E_\lambda \subseteq C^H_{R_0}$ is a minimizer of $\mathcal{A}_\beta(\cdot,E_0,\lambda)$ in $BV(C^H_{R_0}, \{0,1\})$. Hence, by Remark 3.14 E_λ is a minimizer of $\mathcal{A}_\beta(\cdot,E_0,\lambda)$. □

Proposition 3.33 (Strong comparison for minimizers of $\mathcal{B}_\beta$). *Assume that $v_1, v_2 \in L^\infty_{\mathrm{loc}}(\Omega)$, $v_1 > v_2$ a.e. in Ω and $v_2 \ge 0$ a.e. in $\Omega \setminus C$. Suppose also that $\beta_1 \le \beta_2$ satisfy (3.30). Let $u_1, u_2 \in BV(\Omega,[0,1])$ be minimizers of $\mathcal{B}_{\beta_1}(\cdot,v_1,C)$ and $\mathcal{B}_{\beta_2}(\cdot,v_2,C)$ respectively. Then $u_1 \le u_2$ a.e. in Ω.*

Proof. Adding the inequalities $\mathcal{B}_{\beta_1}(u_1,v_1,C) \le \mathcal{B}_{\beta_1}(u_1 \wedge u_2, v_1, C)$ and $\mathcal{B}_{\beta_2}(u_2,v_2,C) \le \mathcal{B}_{\beta_2}(u_1 \vee u_2, v_2, C)$ and using

$$\int_\Omega |D(u_1 \wedge u_2)| + \int_\Omega |D(u_1 \vee u_2)| \le \int_\Omega |Du_1| + \int_\Omega |Du_2|,$$

we establish

$$\int_{\partial\Omega \cap \{u_1 > u_2\}} (\beta_2 - \beta_1)(u_1 - u_2)\, d\mathcal{H}^n \le \int_{\{u_1>u_2\}} (v_2 - v_1)(u_1 - u_2)\, dx.$$

Since $v_1 > v_2$ and $\beta_1 \le \beta_2$, this inequality holds if and only if $|\{u_1 > u_2\}| = 0$, i.e. $u_1 \le u_2$ a.e. in Ω. □

Proposition 3.34 (**Comparison for minimizers of $\mathcal{B}_\beta$**). *Assume that $v_1, v_2 \in L^\infty_{\mathrm{loc}}(\Omega)$, $v_1 \ge v_2$ a.e. in Ω and $v_2 \ge 0$ a.e. in $\Omega \setminus C$. Suppose also that $\beta_1 \le \beta_2$ satisfy (3.30). Then:*

a) there exists a minimizer u_{1*} of $\mathcal{B}_{\beta_1}(\cdot, v_1, C)$ such that $u_{1*} \le u_2$ for any minimizer u_2 of $\mathcal{B}_{\beta_2}(\cdot, v_2, C)$;

b) there exists a minimizer u_2^* of $\mathcal{B}_{\beta_2}(\cdot, v_2, C)$ such that $u_1 \le u_2^*$ for any minimizer u_1 of $\mathcal{B}_{\beta_1}(\cdot, v_1, C)$.

Proof. a) Take $\varepsilon \in (0,1)$. Since $v_1 + \varepsilon > v_2$ a.e. in Ω, by Proposition 3.33 any minimizer $u_1^\varepsilon, u_2 \in BV(\Omega, [0,1])$ of $\mathcal{B}_{\beta_1}(\cdot, v_1+\varepsilon, C)$ and $\mathcal{B}_{\beta_2}(\cdot, v_2, C)$ respectively, satisfies $u_1^\varepsilon \le u_2$. Let $\mathcal{R}_1 := \max\{\mathcal{R}_1(C, v_1), \mathcal{R}_1(C, v_2)\}$. By minimality, $\mathcal{B}_{\beta_1}(u_1^\varepsilon, v_1+\varepsilon, C) \le \mathcal{B}_{\beta_1}(0, v_1+\varepsilon, C) = 0$, and since by Remark 3.31 $u_1^\varepsilon = 0$ a.e. in $\Omega \setminus C^h_{\mathcal{R}_1}$, recalling (3.5) we get

$$\kappa \int_\Omega |Du_1^\varepsilon| \le (\|v_1\|_{L^\infty(C^h_{\mathcal{R}_1})} + 1)|C^h_{\mathcal{R}_1}| < +\infty.$$

By compactness, there exists $u_{1*} \in BV(\Omega, [0,1])$ such that, up to a (not relabelled) subsequence, $u_1^\varepsilon \to u_{1*}$ in $L^1(\Omega)$ and a.e. in Ω as $\varepsilon \to 0^+$. Then any minimizer u_2 of $\mathcal{B}_{\beta_2}(\cdot, v_2, C)$ satisfies $u_{1*} \le u_2$ a.e. in Ω.

It remains to show that u_{1*} is a minimizer of $\mathcal{B}_{\beta_1}(\cdot, v_1, C)$. By (3.65) we may consider only those $u \in BV(\Omega, [0,1])$ with $u = 0$ a.e. in $\Omega \setminus C^h_{\mathcal{R}_1}$ as a competitor. In this case, the continuity of $u \mapsto \int_{C^h_{\mathcal{R}_1}} uv\,dx$, the minimality of u_1^ε and the lower semicontinuity of $\mathtt{C}_\beta(\cdot, \Omega)$ imply

$$\begin{aligned}
\mathcal{B}_{\beta_1}(u, v_1, C) = \lim_{\varepsilon \to 0^+} \mathcal{B}_{\beta_1}(u, v_1+\varepsilon, C) &\ge \liminf_{\varepsilon \to 0^+} \mathcal{B}_{\beta_1}(u_1^\varepsilon, v_1+\varepsilon, C) \\
&\ge \liminf_{\varepsilon \to 0^+} \mathtt{C}_{\beta_1}(u_1^\varepsilon, \Omega) + \lim_{\varepsilon \to 0^+} \int_{C^h_{\mathcal{R}_1}} u_1^\varepsilon(v_1+\varepsilon)\,dx \\
&\ge \mathtt{C}_{\beta_1}(u_{1*}, \Omega) + \int_{C^h_{\mathcal{R}_1}} u_{1*}v_1\,dx = \mathcal{B}_{\beta_1}(u_{1*}, v_1, C).
\end{aligned}$$

b) can be proven in a similar manner. □

Proof of Theorem 3.27. Let $R := \max\{R(E_0), R(F_0)\}$, where $R(E_0)$ and $R(F_0)$ are defined as in (3.31). Then by Theorem 3.12 any minimizer E_λ (resp. F_λ) of $\mathcal{A}_{\beta_1}(\cdot, E_0, \lambda)$ (resp. $\mathcal{A}_{\beta_2}(\cdot, F_0, \lambda)$) is contained in the cylinder $C := \hat{B}_R \times (0, H)$, where

$$H = 1 + \max\Big\{ \max_{(x', x_{n+1}) \in \overline{E_0}} x_{n+1}, \max_{(x', x_{n+1}) \in \overline{F_0}} x_{n+1} \Big\}.$$

Set $v_1 := v_1(\lambda, E_0) = \lambda \tilde{d}_{E_0}$ and $v_2 := v_2(\lambda, F_0) = \lambda \tilde{d}_{F_0}$. Since $E_0 \subseteq F_0 \subset \Omega$, we have $\tilde{d}_{E_0} \ge \tilde{d}_{F_0}$. Moreover, by (3.30) there exists a cylinder $C := C^H_D$ such that $v_2 \ge 0$ in $\Omega \setminus C$.

a) Since $v_1 \ge v_2$ and $\beta_1 \le \beta_2$, by Proposition 3.34 b) there exists a minimizer $u_2^* := u_2^*(\lambda, F_0)$ of $\mathcal{B}_{\beta_2}(\cdot, v_2, C)$ such that any minimizer u_1 of $\mathcal{B}_{\beta_1}(\cdot, v_1, C)$ satisfies

$$u_1 \le u_2^*. \tag{3.67}$$

By Remark 3.31 there exists $t \in (0,1)$ such that $\chi_{\{u_2^*>t\}}$ is a minimizer of $\mathcal{B}_{\beta_2}(\cdot, v_2, C)$. Then, recalling the expression of v_2, by Lemma 3.32 $F_\lambda^* := \{u_2^* > t\}$ is a minimizer of $\mathcal{A}_{\beta_2}(\cdot, F_0, \lambda)$. Moreover, if E_λ is a minimizer of $\mathcal{A}_{\beta_1}(\cdot, E_0, \lambda)$, then by Lemma 3.32 χ_{E_λ} is a minimizer of $\mathcal{B}_{\beta_1}(\cdot, v_1, C)$, and by (3.67) $\chi_{E_\lambda} \le u_2^*$. In particular,

$$E_\lambda = \{\chi_{E_\lambda} > t\} \subseteq \{u_2^* > t\} =: F_\lambda^*.$$

b) is analogous to a) using Proposition 3.34 a).

The last assertion follows with the same arguments from Lemma 3.32 and Proposition 3.33, since (3.62) implies that $\tilde{d}_{E_0} > \tilde{d}_{F_0}$. □

One useful case is when E_0 is a constrained minimizer of $\mathcal{C}_\beta(\cdot, \Omega)$ in $\mathcal{E}(E_0)$: in this case E_0 acts as a barrier for minimizers of $\mathcal{A}_\beta(\cdot, E_0, \lambda)$.

Proposition 3.35. *Assume that E_0, β_1, β_2 satisfy (3.30). Let $\beta_1 \le \beta_2$, E_0 be a constrained minimizer of $\mathcal{C}_{\beta_2}(\cdot, \Omega)$ in $\mathcal{E}(E_0)$ and $E_\lambda \in BV(\Omega, \{0,1\})$ be a minimizer of $\mathcal{A}_{\beta_1}(\cdot, E_0, \lambda)$. Then $E_\lambda \subseteq \overline{E_0}$.*

Proof. Comparing E_λ with $E_0 \cap E_\lambda$ we get

$$P(E_\lambda, \Omega) + \lambda \int_{E_\lambda \setminus E_0} \tilde{d}_{E_0}\, dx \le P(E_\lambda \cap E_0, \Omega) + \int_{\partial\Omega} \beta_1\, \chi_{E_\lambda \setminus E_0}\, d\mathcal{H}^n.$$

From the constrained minimality of E_0 we have $\mathcal{C}_{\beta_2}(E_0, \Omega) \le \mathcal{C}_{\beta_2}(E_0 \cup E_\lambda, \Omega)$, i.e.

$$P(E_0, \Omega) \le P(E_0 \cup E_\lambda, \Omega) - \int_{\partial\Omega} \beta_2\, \chi_{E_\lambda \setminus E_0}\, d\mathcal{H}^n.$$

Adding these inequalities we obtain

$$\begin{aligned} P(E_\lambda, \Omega) + P(E_0, \Omega) + \lambda \int_{E_\lambda \setminus E_0} \tilde{d}_{E_0}\, dx \le & P(E_\lambda \cup E_0, \Omega) + P(E_\lambda \cap E_0, \Omega) \\ & + \int_{\partial\Omega} (\beta_1 - \beta_2) \chi_{E_\lambda \setminus E_0}\, d\mathcal{H}^n. \end{aligned}$$

Then the condition $\beta_1 \le \beta_2$ and (1.2) yield that

$$\lambda \int_{E_\lambda \setminus E_0} \tilde{d}_{E_0}\, dx \le 0.$$

Since $\tilde{d}_{E_0} > 0$ outside $\overline{E_0}$, the last inequality is possible only if $|E_\lambda \setminus \overline{E_0}| = 0$, i.e. $E_\lambda \subseteq \overline{E_0}$. □

Proposition 3.35 gives the following monotonicity principle.

Proposition 3.36 (Monotonicity). *Assume that E_0, β satisfy (3.30), E_0 is a constrained minimizer of $\mathcal{C}_\beta(\cdot, \Omega)$ in $\mathcal{E}(E_0)$ such that $|\overline{E_0} \setminus E_0| = 0$ and $E_\alpha \in BV(\Omega, \{0,1\})$ is a minimizer of $\mathcal{A}_\beta(\cdot, E_0, \alpha)$ for $\alpha \ge 1$. Then $E_\lambda \subseteq E_\mu$ for any $1 \le \lambda < \mu$. Moreover, every E_α, $\alpha \ge 1$ is also a constrained minimizer of $\mathcal{C}_\beta(\cdot, \Omega)$ in $\mathcal{E}(E_\alpha)$.*

Proof. Comparison between E_λ and $E_\lambda \cap E_\mu$ gives

$$P(E_\lambda, \Omega) + \lambda \int_{E_\lambda \setminus E_\mu} \tilde{d}_{E_0}\, dx \leq P(E_\lambda \cap E_\mu, \Omega) + \int_{\partial\Omega} \beta\, \chi_{E_\lambda \setminus E_\mu}\, d\mathcal{H}^n.$$

Similarly, for E_μ and $E_\lambda \cup E_\mu$ we have

$$P(E_\mu, \Omega) \leq P(E_\lambda \cup E_\mu, \Omega) + \mu \int_{E_\lambda \setminus E_\mu} \tilde{d}_{E_0}\, dx - \int_{\partial\Omega} \beta\, \chi_{E_\lambda \setminus E_\mu}\, d\mathcal{H}^n.$$

Adding the above inequalities and using (1.2) we obtain

$$(\lambda - \mu) \int_{E_\lambda \setminus E_\mu} \tilde{d}_{E_0}\, dx \leq 0. \tag{3.68}$$

By hypothesis $|\overline{E_0} \setminus E_0| = 0$, according to Proposition 3.35, $E_\lambda, E_\mu \subseteq E_0$, Thus $\tilde{d}_{E_0} \leq 0$ in $E_\lambda \setminus E_\mu$. But since $\lambda < \mu$, (3.68) is possible only if $|E_\lambda \setminus E_\mu| = 0$, i.e. $E_\lambda \subseteq E_\mu$.

To prove the final assertion take any set $E \in \mathcal{E}(E_\alpha)$. Then using $\mathcal{A}_\beta(E_\alpha, E_0, \alpha) \leq \mathcal{A}_\beta(E_\alpha \cap E_0, E_0, \alpha)$, $\alpha \int_{(E_0 \cap E) \setminus E_\alpha} d_{E_0} dx \geq 0$, and $E_\alpha \subseteq E_0 \cap E$, we get

$$\mathcal{C}_\beta(E_\alpha, \Omega) \leq \mathcal{C}_\beta(E_\alpha, \Omega) + \alpha \int_{(E_0 \cap E) \setminus E_\alpha} d_{E_0} dx \leq \mathcal{C}_\beta(E \cap E_0, \Omega).$$

Moreover, since $\mathcal{C}_\beta(E_0, \Omega) \leq \mathcal{C}_\beta(E \cup E_0, \Omega)$, from (1.2) we obtain

$$\mathcal{C}_\beta(E_\alpha, \Omega) + \mathcal{C}_\beta(E_0, \Omega) \leq \mathcal{C}_\beta(E_0 \cap E, \Omega) + \mathcal{C}_\beta(E_0 \cup E, \Omega) \leq \mathcal{C}_\beta(E, \Omega) + \mathcal{C}_\beta(E_0, \Omega),$$

i.e. $\mathcal{C}_\beta(E_\alpha, \Omega) \leq \mathcal{C}_\beta(E, \Omega)$. □

Proposition 3.37 (Comparison between minimizers of $\mathcal{C}_\beta$ and $\mathcal{A}_\beta$). *Suppose that E_0 and β satisfy* (3.30).

a) Let $E^+ \in BV(\Omega, \{0,1\})$ be a constrained minimizer of $\mathcal{C}_\beta(\cdot, \Omega)$ in $\mathcal{E}(E_0)$. Then every minimizer E_λ of $\mathcal{A}_\beta(\cdot, E_0, \lambda)$ satisfies $E_\lambda \subseteq \overline{E^+}$.

b) Let $E^+ \in BV(\Omega, \{0,1\})$ be a bounded constrained minimizer of $\mathcal{C}_\beta(\cdot, \Omega)$ in $\mathcal{E}(E^+)$. Then for every $E_0 \subseteq E^+$ and for every minimizer E_λ of $\mathcal{A}_\beta(\cdot, E_0, \lambda)$ one has $E_\lambda \subseteq \overline{E^+}$. Moreover, E^+ can be chosen such that $|\overline{E^+} \setminus E^+| = 0$.

Proof. a) By Proposition 3.15 E^+ is a constrained minimizer of $\mathcal{C}_\beta(\cdot, \Omega)$ in $\mathcal{E}(E^+)$. Let E^+_λ be the maximal minimizer of $\mathcal{A}_\beta(\cdot, E^+, \lambda)$ (Definition 3.30). By Proposition 3.35 we have $E^+_\lambda \subseteq \overline{E^+}$. Take any minimizer E_λ of $\mathcal{A}_\beta(\cdot, E_0, \lambda)$. Since $E_0 \subseteq E^+$, by Theorem 3.27 a) we have

$$E_\lambda \subseteq E^+_\lambda \subseteq \overline{E^+}.$$

b) The proof of the first part is exactly the same as the proof of a). To prove the second part, we take any $E'_0 \in BV(\Omega, \{0,1\})$ satisfying the hypotheses of Proposition 3.26 and containing E_0. By Theorem 3.15 there exists a constrained minimizer E^+ of $\mathcal{C}_\beta(\cdot, \Omega)$ in $\mathcal{E}(\underline{E'_0})$. In particular, E^+ is bounded, and by Proposition 3.26 $\mathcal{H}^n(\partial E^+) = P(E^+) < +\infty$. Since $\overline{E^+} \setminus E^+ \subseteq \partial E^+$, we have $|\overline{E^+} \setminus E^+| = 0$. □

3.6 Existence of a generalized minimizing movement

Consider the functional $\widehat{\mathcal{A}_\beta} : BV(\Omega, \{0,1\}) \times BV(\Omega, \{0,1\}) \times [1,+\infty) \times \mathbb{Z} \to [-\infty, +\infty]$ given by

$$\widehat{\mathcal{A}_\beta}(F,G,\lambda,k) := \begin{cases} \mathcal{A}_\beta(F,G,\lambda) & \text{if } k > 0, \\ |F\Delta G| & \text{if } k \le 0, \end{cases}$$

where $[x]$ denotes the integer part of $x \in \mathbb{R}$.

For any $k \in \mathbb{N}$ we build the family of sets $E_\lambda(k)$ iteratively as follows: $E_\lambda(0) := E_0$ and $E_\lambda(k),\ k \ge 1,$ is a minimizer of $\widehat{\mathcal{A}_\beta}(\cdot, E_\lambda(k-1), \lambda, k)$ in $BV(\Omega, \{0,1\});$ notice that existence of minimizers follows from Theorem 3.12.

From now on, we omit the dependence on k of $\widehat{\mathcal{A}_\beta},$ and we use the notation $\widehat{\mathcal{A}_\beta}(F,G,\lambda)$.

Theorem 3.38 (Existence). *Let E_0 and β satisfy (3.40). Then $GMM(E_0)$ is nonempty, i.e. there exist a map $t \in [0,+\infty) \mapsto E(t) \in BV(\Omega, \{0,1\})$ and a diverging sequence $\{\lambda_j\} \subset [1,+\infty)$ such that*

$$\lim_{j\to+\infty} |E_{\lambda_j}([\lambda_j t])\Delta E(t)| = 0, \qquad t \in [0,+\infty). \tag{3.69}$$

Moreover, every GMM $t \in [0,+\infty) \mapsto E(t)$ starting from E_0 is contained in a bounded set depending only on E_0 and $\beta,$ and belongs to $C^{1/2}_{\rm loc}((0,+\infty), L^1(\Omega)),$ in the sense that

$$|E(t)\Delta E(t')| \le \theta(n,\kappa)P(E_0)|t-t'|^{1/2} \qquad \text{for all } t,t' > 0,\ |t-t'| < 1, \tag{3.70}$$

where $\theta(n,\kappa) = \frac{C_{n,\kappa}}{\kappa} + 1$ and $C_{n,\kappa}$ is defined in (3.58). If in addition $|\overline{E_0} \setminus E| = 0,$ then (3.70) holds for any $t,t' \ge 0$ with $|t-t'| < 1.$ Finally,

$$\nu_{E_{\lambda_j}([\lambda_j t])}\mathcal{H}^n \llcorner \partial^* E_{\lambda_j}([\lambda_j t]) \overset{w^*}{\rightharpoonup} \nu_{E(t)}\mathcal{H}^n \llcorner \partial^* E(t) \quad \text{for all } t \ge 0 \text{ as } \lambda_j \to +\infty. \tag{3.71}$$

Proof. Given $k \ge 0$ set $d_k(\cdot) := \operatorname{dist}(\cdot, \Omega \cap \partial E_\lambda(k))$. Then for $k \ge 1$ the minimality of $E_\lambda(k)$ entails

$$\mathcal{A}_\beta(E_\lambda(k), E_\lambda(k-1), \lambda) \le \mathcal{A}_\beta(E_\lambda(k-1), E_\lambda(k-1), \lambda),$$

i.e.

$$\mathcal{C}_\beta(E_\lambda(k), \Omega) + \lambda \int_{E_\lambda(k)\Delta E_\lambda(k-1)} d_{k-1}dx \le \mathcal{C}_\beta(E_\lambda(k-1), \Omega). \tag{3.72}$$

In particular, the sequence $k \in \mathbb{N} \cup \{0\} \mapsto \mathcal{C}_\beta(E_\lambda(k), \Omega)$ is nonincreasing and

$$\mathcal{C}_\beta(E_\lambda(k), \Omega) \le \mathcal{C}_\beta(E_\lambda(0), \Omega) = \mathcal{C}_\beta(E_0, \Omega) \le P(E_0). \tag{3.73}$$

Let $t > 0$ and set $k = [\lambda t]$. Then (3.4) yields

$$\kappa P(E_\lambda([\lambda t])) \le \mathcal{C}_\beta(E_\lambda([\lambda t]), \Omega) \le P(E_0). \tag{3.74}$$

Take $t_1, t_2 > 0,\ t_1 < t_2$ and let $\lambda \ge 1$ be large enough that for some $k_0, N \in \mathbb{N},\ N \ge 3$

$$k_0 = [\lambda t_1], \quad k_0 + N - 1 = [\lambda t_2],$$

i.e.

$$\frac{k_0}{\lambda} \le t_1 < \frac{k_0+1}{\lambda} < \ldots < \frac{k_0+N-1}{\lambda} \le t_2 < \frac{k_0+N}{\lambda}.$$

Then

$$\frac{N-2}{\lambda} = \frac{k_0+N-1-(k_0+1)}{\lambda} \le t_2 - t_1. \tag{3.75}$$

Since all $E_\lambda(s),\ s \ge 1$ satisfy uniform density estimates (3.42)-(3.43) (Theorem 3.20), by Proposition 3.25 we have[3]

$$\begin{aligned} |E_\lambda([\lambda t_2])\Delta E_\lambda([\lambda t_1])| = |E_\lambda(k_0+N-1)\Delta E_\lambda(k_0)| \le \sum_{s=k_0}^{k_0+N-2} |E_\lambda(s)\Delta E_\lambda(s+1)| \\ \le C_{n,\kappa}\ell \sum_{s=k_0}^{k_0+N-2} P(E_\lambda(s)) + \frac{1}{\ell}\sum_{s=k_0}^{k_0+N-2} \int_{E_\lambda(s+1)\Delta E_\lambda(s)} d_{E_\lambda(s)}\,dx \end{aligned} \tag{3.76}$$

for any $\ell \in (0, \frac{\gamma(n,\kappa)}{\lambda^{1/2}})$. The first sum can be estimated using (3.74):

$$\sum_{s=k_0}^{k_0+N-2} P(E_\lambda(s)) \le \frac{P(E_0)}{\kappa}(N-1). \tag{3.77}$$

Moreover, for any $s \in \mathbb{N}$, by (3.72)

$$\int_{E_{\lambda_j}(s+1)\Delta E_\lambda(s)} d_{E_\lambda(s)}\,dx \le \frac{1}{\lambda}\Big(\mathcal{C}_\beta(E_\lambda(s),\Omega) - \mathcal{C}_\beta(E_\lambda(s+1),\Omega)\Big),$$

and thus

$$\begin{aligned} \sum_{s=k_0}^{k_0+N-2} \int_{E_\lambda(s+1)\Delta E_\lambda(s)} d_{E_\lambda(s)}\,dx \le & \frac{1}{\lambda}\sum_{s=k_0}^{k_0+N-2}\Big(\mathcal{C}_\beta(E_\lambda(s),\Omega) - \mathcal{C}_\beta(E_\lambda(s+1),\Omega)\Big) \\ = & \frac{1}{\lambda}\Big(\mathcal{C}_\beta(E_\lambda(k_0),\Omega) - \mathcal{C}_\beta(E_\lambda(k_0+N-1),\Omega)\Big). \end{aligned}$$

Using (3.73) and the nonnegativity of $\mathcal{C}_\beta(\cdot,\Omega)$ we get

$$\sum_{s=k_0}^{k_0+N-2} \int_{E_\lambda(s+1)\Delta E_\lambda(s)} d_{E_\lambda(s)}\,dx \le \frac{P(E_0)}{\lambda}. \tag{3.78}$$

Thus, from (3.76), (3.77) and (3.78)

$$|E_\lambda([\lambda t_1])\Delta E_\lambda([\lambda t_2])| \le \frac{C_{n,\kappa}P(E_0)}{\kappa}(N-1)\ell + \frac{P(E_0)}{\lambda \ell}. \tag{3.79}$$

[3] Notice that at this point we use $t_1 > 0$; since a priori we do not know whether E_0 satisfies the density estimates, we cannot start summing from $s = 0 = k_0$.

Now take λ so large that

$$t_2 - t_1 > \frac{1}{\gamma(n,\kappa)^2\,\lambda},$$

so that Proposition 3.25 holds for $\ell = \frac{1}{\lambda|t_2-t_1|^{1/2}}$. From (3.79) and (3.75) we obtain

$$\begin{aligned}\big|E_\lambda([\lambda t_1])\Delta E_\lambda([\lambda t_2])\big| \le & \frac{C_{n,\kappa}P(E_0)}{\kappa}\,\frac{N-2}{\lambda|t_2-t_1|^{1/2}} + \frac{1}{\lambda}\,\frac{C_{n,\kappa}P(E_0)}{\kappa|t_2-t_1|^{1/2}} + P(E_0)\,|t_2-t_1|^{1/2} \\ \le & \theta(n,\kappa)P(E_0)\,|t_2-t_1|^{1/2} + \frac{1}{\lambda}\,\frac{C_{n,\kappa}P(E_0)}{\kappa|t_2-t_1|^{1/2}}.\end{aligned} \tag{3.80}$$

By virtue of Proposition 3.37 b) there exists a constrained minimizer $E^+ \supseteq E_0$ of $\mathcal{C}_\beta(\cdot,\Omega)$ in $\mathcal{E}(E^+)$ such that $|\overline{E^+}\setminus E^+| = 0$ and $E_\lambda(1) \subseteq E^+$. By induction, we can show that $E_\lambda(k) \subseteq E^+$ for all $k \ge 1$. Consider now an arbitrary diverging sequence $\{\lambda_j\}$. Compactness and a diagonal process yield the existence of a subsequence (still denoted by $\{\lambda_j\}$) such that $E_{\lambda_j}([\lambda_j t])$ converges in $L^1(\Omega)$ to a set $E(t)$ for any rational $t \ge 0$ as $j \to +\infty$.

If $t_1, t_2 \in \mathbb{Q}\cap(0,+\infty)$, with $0 < |t_1 - t_2| < 1$, letting $\lambda_j \to +\infty$ in (3.80) we get

$$|E(t_1)\Delta E(t_2)| \le \theta(n,\kappa)P(E_0)|t_2-t_1|^{1/2}. \tag{3.81}$$

By completeness of $L^1(\Omega)$ we can uniquely extend $\{E(t) : t \in \mathbb{Q}\cap(0,+\infty)\}$ to a family $\{E(t) : t \in (0,+\infty)\}$ preserving the Hölder continuity (3.81) in $(0,+\infty)$. Now we show (3.69). If $t = 0$, $E_0 = E_{\lambda_j}(0) \to E(0)$ in $L^1(\Omega)$ as $j \to +\infty$. If $t > 0$, take any $\varepsilon \in (0,1)$ and $t_\varepsilon \in \mathbb{Q}\cap(0,+\infty)$ such that $|t - t_\varepsilon| < \varepsilon$. By the choice of $\{\lambda_j\}$, (3.69) holds for t_ε and thus, using (3.80)-(3.81) we get

$$\begin{aligned}\limsup_{j\to+\infty} |E_{\lambda_j}([\lambda_j t])\Delta E(t)| \le & \limsup_{j\to+\infty} |E_{\lambda_j}([\lambda_j t])\Delta E_{\lambda_j}([\lambda_j t_\varepsilon])| \\ & + \limsup_{j\to+\infty} |E_{\lambda_j}([\lambda_j t_\varepsilon])\Delta E(t_\varepsilon)| + |E(t_\varepsilon)\Delta E(t)| \\ \le & 2\theta(n,\kappa)P(E_0)|t-t_\varepsilon|^{1/2} < 2\theta(n,\kappa)P(E_0)\sqrt{\varepsilon}.\end{aligned}$$

Therefore, letting $\varepsilon \to 0^+$ we get (3.69).

When $|\overline{E_0}\setminus E_0| = 0$, for any $t \in (0,1)$, choosing λ sufficiently large, from (3.80) we obtain

$$\begin{aligned}|E_\lambda([\lambda t])\Delta E(0)| \le & |E_\lambda([\lambda t])\Delta E_\lambda(1)| + |E_\lambda(1)\Delta E_0| \\ \le & \theta(n,\kappa)P(E_0)\Big|t - \frac{1}{\lambda}\Big|^{1/2} + \frac{1}{\lambda}\frac{C_{n,\kappa}P(E_0)}{\kappa|t-\frac{1}{\lambda}|^{1/2}} + |E_\lambda(1)\Delta E_0|.\end{aligned} \tag{3.82}$$

By Lemma 3.17 a) the last term on the right hand side converges to 0 as $\lambda \to +\infty$. Hence letting $\lambda \to +\infty$ in (3.82) we get the $(1/2)$-Hölder continuity of $t \mapsto E(t)$ in $[0,+\infty)$.

Now let us prove (3.71). We need to show that for any $t \in [0,+\infty)$

$$\lim_{j\to+\infty} \int_{\partial^* E_{\lambda_j}([\lambda_j t])} \phi\cdot\nu_{E_{\lambda_j}([\lambda_j t])}\,d\mathcal{H}^n = \int_{\partial^* E(t)} \phi\cdot\nu_{E(t)}\,d\mathcal{H}^n \quad \forall\phi \in C_c(\mathbb{R}^{n+1},\mathbb{R}^{n+1}).$$

If $\phi \in C_c^1(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})$, by the generalized divergence formula (1.3) and by (3.69) we have

$$\begin{aligned}\lim_{j\to+\infty} \int_{\partial^* E_{\lambda_j}([\lambda_j t])} \phi \cdot \nu_{E_{\lambda_j}([\lambda_j t])} \, d\mathcal{H}^n &= \lim_{j\to+\infty} \int_{E_{\lambda_j}([\lambda_j t])} \operatorname{div} \phi \, d\mathcal{H}^n \\ &= \int_{E(t)} \operatorname{div} \phi \, d\mathcal{H}^n = \int_{\partial^* E(t)} \phi \cdot \nu_{E(t)} \, d\mathcal{H}^n.\end{aligned} \tag{3.83}$$

In general, we approximate $\phi \in C_c(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})$ uniformly with $\phi_k \in C_c^1(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})$, $k \geq 1$ and use the previous result.

Finally, if $\{E(t)\}_{t\geq 0} \in GMM(E_0)$, then by construction and Proposition 3.37 b) one has $E_{\lambda_j}([\lambda_j t]) \subseteq E^+$, where $E^+ := E^+(E_0, \beta)$ is a bounded minimizer of $\mathcal{C}_\beta(\cdot, \Omega)$ in $\mathcal{E}(E^+)$; therefore $E(t) \subseteq E^+$ for all $t \geq 0$. □

Definition 3.39 (Maximal and minimal GMM). *Let E_0, β satisfy (3.40), and $\{\lambda_j\}$ be a diverging sequence such that*

$$E^*(t) := \lim_{j\to+\infty} E_{\lambda_j}([\lambda_j t])^* \qquad \forall t \geq 0$$

exist in $L^1(\Omega)$, where $E_{\lambda_j}([\lambda_j t])^*$ is the maximal minimizer of $\mathcal{A}_\beta(\cdot, E_{\lambda_j}([\lambda_j t]-1)^*, \lambda_j)$ with $(E_0)^* := E_0$ (Definition 3.30). We call $E^*(t)$ the maximal GMM associated to the sequence $\{\lambda_j\}$. Analogously,

$$E_*(t) := \lim_{j\to+\infty} E_{\lambda_j}([\lambda_j t])_* \qquad \forall t \geq 0,$$

obtained using the minimal minimizers $E_{\lambda_j}([\lambda_j t])_*$ of $\widehat{\mathcal{A}}_\beta(\cdot, E_{\lambda_j}([\lambda_j t]-1)_*, \lambda_j)$ with $(E_0)_* := E_0$, is called the minimal GMM associated to the sequence $\{\lambda_j\}$.

Observe that if $t \mapsto E(t)$ is any GMM obtained by the sequence $\{\lambda_j\}$, then according to the proof of Theorem 3.38 (possibly passing to nonrelabelled subsequences) there exist the maximal GMM $t \mapsto E^*(t)$ and the minimal GMM $t \mapsto E_*(t)$ associated to $\{\lambda_j\}$. Now by Remark 3.29 one has $E_*(t) \subseteq E(t) \subseteq E^*(t)$ for all $t \geq 0$.

Theorem 3.40 (Comparison principle for maximal and minimal GMM). *Let $E_0, F_0, \beta_1, \beta_2$ satisfy (3.40) with $E_0 \subseteq F_0$ and $\beta_1 \leq \beta_2$. If $E_*(t)$ and $F_*(t)$ are minimal GMMs associated to a sequence $\{\lambda_j\}$, then $E_*(t) \subseteq F_*(t)$ for all $t \geq 0$. Analogously, if $E^*(t)$ and $F^*(t)$ are maximal GMMs associated to $\{\lambda_j'\}$, then $E^*(t) \subseteq F^*(t)$ for all $t \geq 0$.*

Proof. Since $E_0 \subseteq F_0$, and $\beta_1 \leq \beta_2$, by definition of $E_\lambda(k)^*$ and $F_\lambda(k)^*$ (resp. $E_\lambda(k)_*$ and $F_\lambda(k)_*$) and by Theorem 3.27, we have $E_{\lambda *}(k) \subseteq F_{\lambda *}(k)$ (resp. $E_\lambda^*(k) \subseteq F_\lambda^*(k)$) which implies $E_*(t) \subseteq F_*(t)$ (resp. $E^*(t) \subseteq F^*(t)$) for all $t \geq 0$. □

From the proof of Theorem 3.38 and Propositions 3.35 -3.36 we get the following result (compare with [17]), that could be applied, for instance, to E_0 as in Example 3.16.

Theorem 3.41. *Let E_0 be a constrained minimizer of $\mathcal{C}_\beta(\cdot, \Omega)$ in $\mathcal{E}(E_0)$ such that $|\overline{E_0} \setminus E_0| = 0$. Then every maximal (minimal) GMM $t \mapsto E(t)$ starting from E_0 satisfies $E(t) \subseteq E(t')$ provided $t > t' \geq 0$.*

Proof. Applying Propositions 3.35 and 3.36 inductively to maximal minimizers $E_\lambda(k)^*$ of $\widehat{\mathcal{A}}_\beta(\cdot, E_\lambda(k-1)^*, \lambda)$ we get $E_\lambda(k)^* \subseteq E_\lambda(k-1)^*$ for all $k \geq 1$ and $\lambda \geq 1$. Hence, if $t > t' \geq 0$ then $E_\lambda([\lambda t])^* \subseteq E_\lambda([\lambda t'])^*$. Now the assertion of the theorem follows from (3.69). The arguments for minimal minimizers are the same. □

3.7 GMM as a distributional solution

The aim of this section is to prove that under suitable assumptions GMM is in fact a distributional solution of (3)-(4). Let us start with the following

Definition 3.42 (Admissible variation). *A vector field* $X = (X', X_{n+1}) \in C^1_c(\overline{\Omega}, \mathbb{R}^{n+1})$ *is called admissible if* $X \cdot e_{n+1} = 0$ *on* $\partial\Omega$.

Observe that if $X \in C^1_c(\overline{\Omega}, \mathbb{R}^{n+1})$ is admissible, then for any $s \in (-\varepsilon, \varepsilon)$ with $\varepsilon > 0$ sufficiently small, the vector field $f_s = \mathrm{Id} + sX$ is a C^1-diffeomorphism that satisfies $f_s(\Omega) = \Omega$, $f_s(\overline{\Omega}) = \overline{\Omega}$.

Proposition 3.43 (First variation of $\mathcal{A}_\beta$). *Suppose that* E_0, β *satisfy assumptions* (3.40) *and let* $E \in BV(\Omega, \{0,1\})$ *be bounded with* $\mathrm{Tr}(E) \in BV(\mathbb{R}^n, \{0,1\})$. *Then*

$$\begin{aligned} \frac{d}{ds}\Big|_{s=0} \mathcal{A}_\beta(f_s(E), E_0, \lambda) = & \int_{\Omega\cap\partial^* E} (\operatorname{div} X - \nu_E \cdot (\nabla X)\nu_E)\, d\mathcal{H}^n \\ & + \lambda \int_{\Omega\cap\partial^* E} \tilde{d}_{E_0}\, X \cdot \nu_E \, d\mathcal{H}^n - \int_{\partial^* \mathrm{Tr}(E)} \beta\, X' \cdot \nu'_{\mathrm{Tr}(E)} d\mathcal{H}^{n-1}, \end{aligned} \tag{3.84}$$

where $\partial^*\mathrm{Tr}(E)$ *is the essential boundary of* $\mathrm{Tr}(E)$ *on* $\partial\Omega$ *and* $\nu'_{\mathrm{Tr}(E)}$ *is the outer unit normal to* $\mathrm{Tr}(E) \subset \mathbb{R}^n$.

Proof. From [79, Theorem 17.5]

$$\frac{d}{ds}\Big|_{s=0} P(f_s(E), \Omega) = \int_{\Omega\cap\partial^* E} (\operatorname{div} X - \nu_E \cdot (\nabla X)\nu_E)\, d\mathcal{H}^n.$$

Moreover, [79, Theorem 17.8] and the admissibility of X imply that

$$\frac{d}{ds}\Big|_{s=0} \int_{f_s(E)} \tilde{d}_{E_0}\, dx = \int_{\partial^* E} \tilde{d}_{E_0}\, X \cdot \nu_E \, d\mathcal{H}^n = \int_{\Omega\cap\partial^* E} \tilde{d}_{E_0}\, X \cdot \nu_E \, d\mathcal{H}^n.$$

Finally, since $\mathrm{Tr}(E)$ is a set of finite perimeter in $\partial\Omega \equiv \mathbb{R}^n$, again using [79, Theorem 17.8] we get

$$\frac{d}{ds}\Big|_{s=0} \int_{\partial\Omega} \beta\, \chi_{f_s(E)} d\mathcal{H}^n = \int_{\partial^* \mathrm{Tr}(E)} \beta\, X' \cdot \nu'_{\mathrm{Tr}(E)}\, d\mathcal{H}^{n-1}.$$

□

Remark 3.44. Under assumptions (3.40) and $\beta \in \mathrm{Lip}(\partial\Omega)$, if E_λ is a minimizer of $\mathcal{A}_\beta(\cdot, E_0, \lambda)$, and if $\Omega \cap \partial E_\lambda$ is a C^2-manifold with $\mathcal{H}^{n-1}$- rectifiable boundary, then the mean curvature H_{E_λ} of $\Omega \cap \partial E_\lambda$ is equal to $-\lambda \tilde{d}_{E_0}$. Indeed, using the tangential divergence formula for manifolds with boundary we have

$$\int_{\Omega\cap\partial E_\lambda} (\operatorname{div} X - \nu_{E_\lambda} \cdot (\nabla X)\nu_{E_\lambda})\, d\mathcal{H}^n = \int_{\Omega\cap\partial E_\lambda} H_{E_\lambda}\, X \cdot \nu_{E_\lambda}\, d\mathcal{H}^n + \int_{\partial^* \mathrm{Tr}(E_\lambda)} X' \cdot \mathrm{n}^{\lambda'}\, d\mathcal{H}^{n-1},$$

where $\mathrm{n}^\lambda = (\mathrm{n}^{\lambda'}, \mathrm{n}^\lambda_{n+1})$ is the outer unit conormal to $\overline{\Omega\cap\partial E_\lambda}$ at $\overline{\Omega\cap\partial E_\lambda}\cap\partial\Omega$. By minimality of E_λ, we have $\frac{d}{ds}\mathcal{A}_\beta(f_s(E_\lambda), E_0, \lambda) = 0$, i.e.

$$\int_{\Omega\cap\partial E_\lambda} (H_{E_\lambda} + \lambda\tilde{d}_{E_0})\, X\cdot\nu_{E_\lambda}\, d\mathcal{H}^n + \int_{\partial^*\mathrm{Tr}(E_\lambda)} X'\cdot(\mathrm{n}^{\lambda'} - \beta\nu'_{\mathrm{Tr}(E_\lambda)})\, d\mathcal{H}^{n-1} = 0.$$

This implies $H_{E_\lambda} = -\lambda\tilde{d}_{E_0}$ and $\mathrm{n}^{\lambda'} = \beta\nu'_{\mathrm{Tr}(E_\lambda)}$. Notice that from the latter in particular, we get

$$\beta = \mathrm{n}^\lambda\cdot(\nu'_{\mathrm{Tr}(E_\lambda)}, 0) = \nu_{E_\lambda}\cdot e_{n+1},$$

accordingly for instance with Theorem 3.22.

Remark 3.44 motivates the following definition [14, 79].

Definition 3.45 (Distributional mean curvature). *Let $E\in BV(\Omega,\{0,1\})$. The function $H_E \in L^1(\Omega\cap\partial^*E; \mathcal{H}^n \llcorner(\Omega\cap\partial^*E))$ is called distributional mean curvature of $\Omega\cap\partial^*E$ if for every $X\in C^1_c(\Omega,\mathbb{R}^{n+1})$ the generalized tangential divergence formula holds:*

$$\int_{\Omega\cap\partial^*E} (\operatorname{div} X - \nu_E\cdot(\nabla X)\nu_E)\; d\mathcal{H}^n = \int_{\Omega\cap\partial^*E} H_E\, X\cdot\nu_E\, d\mathcal{H}^n. \tag{3.85}$$

Given $x\in\mathbb{R}^{n+1}$ and $t>0$ set

$$v_\lambda(t,x) := \begin{cases} -\lambda\tilde{d}_{E_\lambda([\lambda t]-1)}(x) & \text{if } t\ge\frac{1}{\lambda},\\ 0 & \text{if } t\in[0,\frac{1}{\lambda}). \end{cases}$$

Remark 3.46. By Theorem 3.22, $\mathrm{Tr}(E_\lambda([\lambda t]))\in BV(\mathbb{R}^n,\{0,1\})$.

The next result relates GMM with distributional solutions of (3)-(4).

Theorem 3.47 (GMM is a distributional solution). *Let E_0,β satisfy (3.40), $|\overline{E_0}\setminus E_0| = 0$, $\{E(t)\}_{t\ge0}$ be a GMM starting from E_0 obtained along the diverging sequence $\{\lambda_j\}$. Suppose that*

$$\mathcal{H}^n\llcorner(\Omega\cap\partial^*E_{\lambda_j}([\lambda_j t])) \overset{w^*}{\rightharpoonup} \mathcal{H}^n\llcorner(\Omega\cap\partial^*E(t)) \quad \text{as } j\to+\infty \text{ for a.e. } t\ge0. \tag{3.86}$$

Then there exist a function $v:[0,+\infty)\times\Omega\to\mathbb{R}$ with

$$\int_0^{+\infty}\int_{\Omega\cap\partial^*E(t)} (v)^2\, d\mathcal{H}^n\, dt \le \alpha(n,\kappa)\, P(E_0), \tag{3.87}$$

and a (not relabelled) subsequence such that

$$\lim_{j\to+\infty}\int_0^{+\infty}\int_{\Omega\cap\partial^*E_{\lambda_j}([\lambda_j t])} \phi v_{\lambda_j}\, d\mathcal{H}^n dt = \int_0^{+\infty}\int_{\Omega\cap\partial^*E(t)} \phi v\, d\mathcal{H}^n dt, \tag{3.88}$$

$$\lim_{j\to+\infty}\int_0^{+\infty}\int_{\Omega\cap\partial^*E_{\lambda_j}([\lambda_j t])} v_{\lambda_j}\,\nu_{E_{\lambda_j}([\lambda_j t])}\cdot\Psi\, d\mathcal{H}^n dt = \int_0^{+\infty}\int_{\Omega\cap\partial^*E(t)} v\,\nu_{E(t)}\cdot\Psi\, d\mathcal{H}^n dt \tag{3.89}$$

for any $\phi\in C_c(\Omega)$, $\Psi\in C_c([0,+\infty)\times\Omega,\mathbb{R}^{n+1})$, where $\alpha(n,\kappa) := \frac{75[(n+1)\omega_{n+1}+\omega_n]\mathfrak{b}(n)}{(\kappa/2)^{n+1}\omega_{n+1}}$. Moreover, $\{E(t)\}_{t\ge0}$ solves (3)-(4) with initial datum E_0 in the following sense:

(i) *for a.e.* $t \geq 0$ *the set* $\Omega \cap \partial^* E(t)$ *has distributional mean curvature* $H_{E(t)} = v$ *and if* $1 \leq n \leq 6$, *for every* $\phi \in C_c^1([0,+\infty) \times \Omega)$:

$$\int_0^{+\infty} \int_{E(t)} \partial_t \phi \, dx dt + \int_{E(0)} \phi(0,x)\, dx = \int_0^{+\infty} \int_{\Omega\cap\partial^* E(t)} \phi H_{E(t)} \, d\mathcal{H}^n dt; \tag{3.90}$$

(ii) *if* $\beta \in \mathrm{Lip}(\partial\Omega)$ *and there exists* $h \in L^1_{\mathrm{loc}}([0,+\infty))$ *such that*

$$P(\mathrm{Tr}(E_{\lambda_j}([\lambda_j t]))) \leq h(t) \qquad \textit{for all } j \geq 1 \textit{ and a.e. } t \geq 0, \tag{3.91}$$

then $\mathrm{Tr}(E(t)) \in BV(\mathbb{R}^n, \{0,1\})$ *for a.e.* $t > 0$ *and*

$$\begin{aligned} &\int_{\Omega\cap\partial^* E(t)} \Big(\mathrm{div}\, X - \nu_{E(t)} \cdot (\nabla X) \nu_{E(t)} \Big) \, d\mathcal{H}^n \\ &\qquad = \int_{\Omega\cap\partial^* E(t)} H_{E(t)} \, X \cdot \nu_{E(t)} \, d\mathcal{H}^n + \int_{\partial^*\mathrm{Tr}(E(t))} \beta \, X' \cdot \nu'_{\mathrm{Tr}(E(t))} \, d\mathcal{H}^{n-1} \end{aligned} \tag{3.92}$$

for every admissible $X \in C_c^1(\overline{\Omega}, \mathbb{R}^{n+1})$.

The need for assumption (3.86) is not surprising; see [78, 91] for conditional results obtained in other contexts in a similar spirit. We postpone the proof after several auxiliary results.

Proposition 3.48. *Assume that* E_0 *and* β *satisfy* (3.40). *Then for any* $\lambda \geq 1$ *and a.e.* $t \geq 1/\lambda$ *the function* $v_\lambda(t,\cdot)$ *is the distributional mean curvature of* $E_\lambda([\lambda t])$.

Proof. Set $E := E_\lambda([\lambda t])$. Remark 3.46 and (3.84) imply that

$$\int_{\Omega\cap\partial^* E} (\mathrm{div}\, X - \nu_E \cdot (\nabla X)\nu_E) \, d\mathcal{H}^n = \int_{\Omega\cap\partial^* E} v_\lambda \, X \cdot \nu_E \, d\mathcal{H}^n.$$

Hence, it suffices to prove $v_\lambda(t,\cdot) \in L^1(\Omega\cap\partial^* E; \mathcal{H}^n \llcorner \Omega \cap \partial^* E)$ for a.e. $t \in [1/\lambda, +\infty)$ and since $P(E(t),\Omega) < +\infty$, this follows from Lemma 3.50 below. □

Remark 3.49. From Definition 3.45, Proposition 3.48 and Lemma 3.50 it follows that

$$v_\lambda(t,x) = H_{E_\lambda([\lambda t])}(t,x) \quad \text{for a.e. } t \geq 1/\lambda \text{ and } \mathcal{H}^n\text{-a.e. } x \in \Omega \cap \partial E_\lambda([\lambda t]).$$

This is a discretized version of equation (3).

Lemma 3.50 (Uniform L^2-bound of the approximate velocities). *Under assumptions* (3.40) *the inequality*

$$\int_0^{+\infty} \int_{\Omega\cap\partial E_\lambda([\lambda t])} (v_\lambda)^2 \, d\mathcal{H}^n dt \leq \alpha(n,\kappa) P(E_0)$$

holds.

Proof. The proof is analogous to the proof of [91, Lemma 3.6]. Given $\varepsilon > 0$ and $E \in BV(\Omega, \{0,1\})$ let

$$(\partial E)^+_\varepsilon := \{x \in \mathbb{R}^{n+1} : \ \mathrm{dist}(x, \Omega \cap \partial E) \le \varepsilon\}.$$

For $t \in [\frac{1}{\lambda}, +\infty)$ and $\ell \in \mathbb{Z}$ such that $\ell \le 1 + [\log_2(R(n,\kappa)\lambda^{1/2})]$, where $R(n,\kappa)$ is given by (3.41), define

$$K(\ell) = \Big\{x \in \Big(\partial E_\lambda([\lambda t]-1)\Big)^+_{R(n,\kappa)\lambda^{-1/2}} : \ 2^\ell < |v_\lambda(x,t)| \le 2^{\ell+1}\Big\}.$$

By Proposition 3.24 $E_\lambda([\lambda t]) \Delta E_\lambda([\lambda t]-1) \subseteq \cup_\ell K(\ell)$. Take $x \in K(\ell) \cap \Omega \cap \partial E_\lambda([\lambda t])$. Then $B_{\frac{2^{\ell-1}}{\lambda}}(x) \cap E_\lambda([\lambda t]-1) = \emptyset$ and hence, by Remark 3.23 the following density estimates hold:

$$\begin{aligned} |E_\lambda([\lambda t]) \cap B_{\frac{2^{\ell-1}}{\lambda}}(x)| &\ge \Big(\frac{\kappa}{2}\Big)^{n+1} \omega_{n+1} \Big(\frac{2^{\ell-1}}{\lambda}\Big)^{n+1}, \\ \mathcal{H}^n(B_{\frac{2^{\ell-1}}{\lambda}}(x) \cap \Omega \cap \partial E_\lambda([\lambda t])) &\le \Big[(n+1)\omega_{n+1} + \omega_n\Big] \Big(\frac{2^{\ell-1}}{\lambda}\Big)^n. \end{aligned} \tag{3.93}$$

Using $2^{\ell-1} \le |v_\lambda(y,t)| \le 5 \cdot 2^{\ell-1}$ for any $y \in B_{\frac{2^{\ell-1}}{\lambda}}(x)$, from (3.93) we deduce

$$\begin{aligned} \int_{B_{\frac{2^{\ell-1}}{\lambda}}(x) \cap \Omega \cap \partial E_\lambda([\lambda t])} (v_\lambda)^2 \, d\mathcal{H}^n \le& 25[(n+1)\omega_{n+1} + \omega_n](2^{\ell-1})^2 \Big(\frac{2^{\ell-1}}{\lambda}\Big)^n \\ \le& \frac{25[(n+1)\omega_{n+1} + \omega_n]}{(\kappa/2)^{n+1}\omega_{n+1}} \lambda \int_{B_{\frac{2^{\ell-1}}{\lambda}}(x) \cap (E_\lambda([\lambda t]) \Delta E_\lambda([\lambda t]-1))} |v_\lambda| \, dx. \end{aligned}$$

Application of the Besicovitch covering theorem to the collection of balls $\{B_{\frac{2^{\ell-1}}{\lambda}}(x) : \ x \in K(\ell) \cap \partial E_\lambda([\lambda t])\}$ gives

$$\int_{K(\ell) \cap \Omega \cap \partial E_\lambda([\lambda t])} (v_\lambda)^2 \, d\mathcal{H}^n \le \frac{25[(n+1)\omega_{n+1} + \omega_n]\mathfrak{b}(n)}{(\kappa/2)^{n+1}\omega_{n+1}} \lambda \int_{\{2^{\ell-1} \le |v_\lambda| \le 2^{\ell+2}\} \cap (E_\lambda([\lambda t]) \Delta E_\lambda([\lambda t]-1))} |v_\lambda| \, dx.$$

Now summing up these inequalities over $\ell \in \mathbb{Z}$ with $\ell \le 1 + [\log_2(R(n,\kappa)\lambda^{1/2})]$, and using the properties of $K(\ell)$ and the definition of $\alpha(n,\kappa)$ we get

$$\int_{\Omega \cap \partial E_\lambda([\lambda t])} (v_\lambda)^2 \, d\mathcal{H}^n \le \alpha(n,\kappa) \, \lambda \int_{E_\lambda([\lambda t]) \Delta E_\lambda([\lambda t]-1)} |v_\lambda| \, dx.$$

Observe that by (3.72) for any $t \ge 1/\lambda$ one has

$$\int_{E_\lambda([\lambda t]) \Delta E_\lambda([\lambda t]-1)} |v_\lambda| \, dx \le \mathcal{C}_\beta(E_\lambda([\lambda t]-1), \Omega) - \mathcal{C}_\beta(E_\lambda([\lambda t]), \Omega).$$

Thus

$$\int_{\Omega \cap \partial E_\lambda([\lambda t])} (v_\lambda)^2 \, d\mathcal{H}^n \le \alpha(n,\kappa) \, \lambda \Big(\mathcal{C}_\beta(E_\lambda([\lambda t]-1), \Omega) - \mathcal{C}_\beta(E_\lambda([\lambda t]), \Omega)\Big).$$

Fixing $T > 0$ and integrating this inequality in $t \in [0,T]$ we get

$$\begin{aligned}\int_0^T \int_{\Omega\cap\partial E_\lambda([\lambda t])} (v_\lambda)^2 \, d\mathcal{H}^n dt \le & \alpha(n,\kappa) \sum_{k=1}^{[T\lambda]+1} \Big(\mathcal{C}_\beta(E_\lambda(k-1),\Omega) - \mathcal{C}_\beta(E_\lambda(k),\Omega)\Big) \\ \le & \alpha(n,\kappa)\, \mathcal{C}_\beta(E_0,\Omega) \le \alpha(n,\kappa)\, P(E_0),\end{aligned}$$

where we used (3.4). Now letting $T \to +\infty$ completes the proof. □

Proposition 3.51. *Let* E_0, β *satisfy* (3.40)*,* $\lambda \ge 1$ *and* E^+ *be as in Proposition 3.37. Then*

$$\lambda \int_{1/\lambda}^T |E_\lambda([\lambda t])\Delta E_\lambda([\lambda t]-1)|\, dt \le |E^+| + \frac{P(E_0)}{\gamma(n,\kappa)} + \frac{2^{n+1}\omega_{n+1}\gamma(n,\kappa)\mathfrak{b}(n)}{\kappa c(n,\kappa)} P(E_0)\, T \quad (3.94)$$

for any $T > \frac{1}{\lambda}$*. Here* $\mathfrak{b}(n), \gamma(n,\kappa), c(n,\kappa)$ *are defined in Section 3.4.*

Proof. Let $[\lambda T] = N$. Clearly,

$$\lambda \int_{1/\lambda}^T |E_\lambda([\lambda t])\Delta E_\lambda([\lambda t]-1)|\, dt = \sum_{k=1}^N |E_\lambda(k)\Delta E_\lambda(k-1)|.$$

We recall that $E_\lambda(k) \subset E^+$ for all $\lambda \ge 1$ and $k \ge 0$, by Proposition 3.37.

If $k = 1$, then

$$|E_\lambda(1)\Delta E_\lambda(0)| \le |E^+|. \quad (3.95)$$

Now if $k \ge 2$, we write $E_\lambda(k)\Delta E_\lambda(k-1)$ as a union of A_k and B_k, where

$$A_k = \Big\{x \in E_\lambda(k)\Delta E_\lambda(k-1) : \; d_{E_\lambda(k-1)}(x) > \ell\Big\},$$

$$B_k = \Big\{x \in E_\lambda(k)\Delta E_\lambda(k-1) : \; d_{E_\lambda(k-1)}(x) \le \ell\Big\}.$$

where $\ell := \frac{\gamma(n,\kappa)}{\lambda}$. By Chebyshev inequality $|A_k|$ can be estimated using (3.72) as

$$|A_k| \le \frac{\lambda}{\gamma(n,\kappa)} \int_{E_\lambda(k)\Delta E_\lambda(k-1)} d_{E_\lambda(k-1)}\, dx \le \frac{1}{\gamma(n,\kappa)} \Big(\mathcal{C}_\beta(E_\lambda(k),\Omega) - \mathcal{C}_\beta(E_\lambda(k),\Omega)\Big).$$

Hence, by (3.74)

$$\sum_{k=2}^N |A_k| \le \frac{1}{\gamma(n,\kappa)} \sum_{k=2}^N \Big(\mathcal{C}_\beta(E_\lambda(k),\Omega) - \mathcal{C}_\beta(E_\lambda(k),\Omega)\Big) \le \frac{P(E_0)}{\gamma(n,\kappa)}.$$

Moreover, by definition B_k can be covered by the family of balls $\{B_{2\ell}(x),\, x \in \partial E_\lambda(k-1)\}$. Thus, by Besicovitch covering theorem we can find at most countably many balls $\{B_\ell(x_j),\, x_j \in \partial E_\lambda(k-1)\}$ covering $\Omega \cap \partial E_\lambda(k-1)$. Hence, the lower density estimate (3.43) for $E_\lambda(k-1)$ used with ℓ implies

$$|B_{2\ell}(x_j) \cap B_k| \le (2^{n+1}\omega_{n+1}\ell)\ell^n \le \frac{2^{n+1}\omega_{n+1}}{c(n,\kappa)}\, \ell\, P(E_\lambda(k-1), B_\ell(x_j)),$$

from which it follows that

$$\sum_{k=2}^{N}|B_k| \le \sum_{k=2}^{N}\sum_{j\ge1}|B_{2\ell}(x_j)\cap B_k| \le \frac{2^{n+1}\omega_{n+1}}{c(n,\kappa)}\,\ell\sum_{k=2}^{N}\sum_{j\ge1}P(E_\lambda(k-1),B_\ell(x_j))$$

$$\le \frac{2^{n+1}\mathfrak{b}(n)\omega_{n+1}}{c(n,\kappa)}\,\ell\sum_{k=2}^{N}P(E_\lambda(k-1),\Omega).$$

Therefore, using (3.74) and $N\le\lambda T,$ we get

$$\sum_{k=2}^{N}|B_k| \le \frac{2^{n+1}\mathfrak{b}(n)\omega_{n+1}\gamma(n,\kappa)}{\kappa c(n,\kappa)}\,P(E_0)\,T. \tag{3.96}$$

Finally, (3.94) follows from (3.95)-(3.96). □

The following proposition is the error estimate obtained following [78, 91].

Proposition 3.52 (Error estimate). *Let* $1\le n\le 6.$ *Under assumption* (3.30) *for every* $\phi\in C_c([0,+\infty)\times\Omega)$ *the following error-estimate holds:*

$$\lim_{j\to+\infty}\int_{1/\lambda_j}^{+\infty}\lambda_j\left(\int_\Omega(\chi_{E_{\lambda_j}([\lambda_j t])}-\chi_{E_{\lambda_j}([\lambda_j t]-1)})\phi\,dx-\int_{\Omega\cap\partial E_{\lambda_j}([\lambda_j t])}\tilde d_{E_{\lambda_j}([\lambda_j t]-1)}\,\phi\,d\mathcal{H}^n\right)dt\to0. \tag{3.97}$$

Proof. Given an integer $k\ge1$ set

$$R_k(j):=\int_{k/\lambda_j}^{(k+1)/\lambda_j}\lambda_j\left(\int_\Omega(\chi_{E_{\lambda_j}([\lambda_j t])}-\chi_{E_{\lambda_j}([\lambda_j t]-1)})\,\phi\,dx-\int_{\Omega\cap\partial E_{\lambda_j}([\lambda_j t])}\tilde d_{E_{\lambda_j}([\lambda_j t]-1)}\,\phi\,d\mathcal{H}^n\right)dt.$$

Let us assume that $\operatorname{supp}\phi\subset\subset[0,T)\times\Omega^\varepsilon,\ \ \Omega^\varepsilon:=\mathbb{R}^n\times(\varepsilon,+\infty)\subset\Omega$ for some $\varepsilon,T>0.$ Let us take j so large that $2R\lambda_j^{-1/2}<\varepsilon,$ where $R:=R(n,\kappa)$ is defined in (3.31).

We need to estimate

$$\int_{1/\lambda_j}^{T}\lambda_j\left(\int_{\Omega^\varepsilon}(\chi_{E_{\lambda_j}([\lambda_j t])}-\chi_{E_{\lambda_j}([\lambda_j t]-1)})\,\phi\,dx-\int_{\Omega^\varepsilon\cap\partial E_{\lambda_j}([\lambda_j t])}\tilde d_{E_{\lambda_j}([\lambda_j t]-1)}\,\phi\,d\mathcal{H}^n\right)dt=\sum_{k=1}^{N}R_k(j),$$

where $N=[\lambda_j T].$ Observe that from Proposition 3.24 and (3.74) we get

$$|R_1(j)|:=\left|\int_{1/\lambda_j}^{2/\lambda_j}\lambda_j\left(\int_{\Omega^\varepsilon}(\chi_{E_{\lambda_j}([\lambda_j t])}-\chi_{E_{\lambda_j}([\lambda_j t]-1)})\,\phi\,dx-\int_{\Omega^\varepsilon\cap\partial E_{\lambda_j}([\lambda_j t])}\tilde d_{E_{\lambda_j}([\lambda_j t]-1)}\,\phi\,d\mathcal{H}^n\right)dt\right|$$

$$\le\|\phi\|_\infty\left(|E_{\lambda_j}(1)\Delta E_0|+\frac{R(n,\kappa)}{\sqrt{\lambda_j}}\,P(E_{\lambda_j}(1),\Omega)\right)$$

$$\le\|\phi\|_\infty\left(|E_{\lambda_j}(1)\Delta E_0|+\frac{R(n,\kappa)}{\kappa\sqrt{\lambda_j}}\,P(E_0)\right).$$

Hence, by Lemma 3.17 $R_1(j)\to 0$ as $j\to+\infty$. To estimate $R_k(j)$ for $k\ge 2$ we split it into several steps.

Step 1. Let us show that if $\frac12<\sigma_1<\sigma_2<1,$ and $x\in\Omega^\varepsilon\cap\partial E_{\lambda_j}(k)$ is such that

$$d_{E_{\lambda_j}(k-1)}(y)\le\lambda_j^{-\sigma_2},\quad \forall y\in B_{R\lambda_j^{-1/2}}(x)\cap(E_{\lambda_j}(k)\Delta E_{\lambda_j}(k-1)),$$

then there exists $\nu\in\mathbb{S}^n$ and a continuous increasing function $w\in C([0,\infty))$ with $w(0)=0$ such that

$$\begin{aligned}|\nu_{\partial E_{\lambda_j}(k)}(y)-\nu|&\le w(1/\lambda_j)\quad \forall y\in B_{\lambda_j^{-\sigma_1}}(x)\cap\partial E_{\lambda_j}(k),\\ |\nu_{\partial E_{\lambda_j}(k-1)}(y)-\nu|&\le w(1/\lambda_j)\quad \forall y\in B_{\lambda_j^{-\sigma_1}}(x)\cap\partial E_{\lambda_j}(k-1).\end{aligned}\tag{3.98}$$

The claim is proved exactly the same as [91, Proposition 4.2.1] using the blow-up argument so we give only few details. Let $r\in(0,\lambda_j^{\sigma_1-1/2})$ be arbitrary. Thanks to the choice of ϕ the minimality of $E_{\lambda_j}(k)$ in the ball $B_{r\lambda_j^{-\sigma_1}}(x)$ reads as

$$P(E_{\lambda_j}(k),B_{r\lambda_j^{-\sigma_1}}(x))\le P(F,B_{r\lambda_j^{-\sigma_1}}(x))+\lambda_j\int_{F\Delta E_{\lambda_j}(k)}d_{E_{\lambda_j}(k-1)}dy$$

for every $F\in BV(\Omega,\{0,1\})$ with $F\Delta E_{\lambda_j}(k)\subset\subset B_{r\lambda_j^{-\sigma_1}}(x),$ since the ball $B_{r\lambda_j^{-\sigma_1}}(x)$ does not intersect $\partial\Omega$. Clearly,

$$d_{E_{\lambda_j}(k-1)}(y)\le r\lambda_j^{-\sigma_1}+R\lambda_j^{-1/2}\le\frac{R+1}{\lambda_j^{1/2}},\quad y\in B_{r\lambda_j^{-\sigma_1}}(x)\cap(F\Delta E_{\lambda_j}(k)),$$

and hence,

$$\lambda_j\int_{F\Delta E_{\lambda_j}(k)}d_{E_{\lambda_j}(k-1)}dy\le(R+1)\lambda_j^{-1/2}|F\Delta E_{\lambda_j}(k)|.$$

Therefore,

$$P(E_{\lambda_j}(k),B_{r\lambda_j^{-\sigma_1}}(x))\le P(F,B_{r\lambda_j^{-\sigma_1}}(x))+(R+1)\lambda_j^{-1/2}|F\Delta E_{\lambda_j}(k)|.\tag{3.99}$$

Similar inequality holds also for $E_{\lambda_j}(k-1)$ since $k\ge2$. Now introduce the blow-up sets

$$E_{\lambda_j}^{\sigma_1}(k):=\Big\{z\in\mathbb{R}^{n+1}:\ z=\frac{y-x}{\lambda_j^{\sigma_1}},\ y\in E_{\lambda_j}(k)\Big\},$$

$$E_{\lambda_j}^{\sigma_1}(k-1):=\Big\{z\in\mathbb{R}^{n+1}:\ z=\frac{y-x}{\lambda_j^{\sigma_1}},\ y\in E_{\lambda_j}(k-1)\Big\}.$$

Using blow-ups from (3.99) we obtain

$$P(E_{\lambda_j}^{\sigma_1}(s),B_r)\le P(F,B_r)+\lambda_j^{\frac12-\sigma_1}(R+1)|E_{\lambda_j}^{\sigma_1}(s)\Delta F|$$

for any $r\in(0,\lambda_j^{\sigma_1-1/2})$ and $F\Delta E_{\lambda_j}^{\sigma_1}(s)\subset\subset B_r(0),$ $s=k,k-1.$

From here we conclude that $E_{\lambda_j}^{\sigma_1}(s),\ \ s=k,k-1$ are $(\lambda_j^{\frac12-\sigma_1}(R+1),\lambda_j^{\sigma_1-1/2})$-minimizer of the perimeter (see [79, Section 23]). Hence, by compactness [79, Proposition 23.13] up to the subsequence,

$$E_{\lambda_j}^{\sigma_1}(s)\to E_s^{\sigma_1}\quad \text{in } L^1(\mathbb{R}^{n+1}) \text{ as } j\to+\infty,\ \ s=k,k-1.$$

In particular, the condition $\sigma_1>1/2$ implies the local minimality of $E_s^{\sigma_1},\ \ s=k,k-1.$ Since $n\le 6,$ by [63, Theorem 17.3] $E_k^{\sigma_1}$ and $E_{k-1}^{\sigma_1}$ half-spaces. Moreover, by hypothesis

$$d_{E_{\lambda_j}^{\sigma_1}(k-1)}(z)\le \lambda_j^{\sigma_1-\sigma_2}\quad \forall z\in B_{\lambda_j^{\beta-1/2}}(0)\cap\big(E_{\lambda_j}^{\sigma_1}(k)\Delta E_{\lambda_j}^{\sigma_1}(k-1)\big),$$

and, therefore, $E_k^{\sigma_1}=E_{k-1}^{\sigma_1},$ i.e. there exists $\nu\in\mathbb{S}^n$ such that

$$E_k^{\sigma_1}=E_{k-1}^{\sigma_1}=\{z\in\mathbb{R}^{n+1}:\ z\cdot\nu<0\}.$$

By [79, Theorem 26.6] $\nu_{E_s^{\sigma_1}}\to\nu$ uniformly in $B_1(0),$ which implies (3.98).

Step 2. According to [91, Corollary 4.2.2] if $\frac12<\sigma_1<\sigma_2<1,$ and $x\in\Omega^\varepsilon\cap\partial E_{\lambda_j}(k)$ is such that

$$d_{E_{\lambda_j}(k-1)}(y)\le\lambda_j^{-\sigma_2},\quad \forall y\in B_{R\lambda_j^{-1/2}}(x)\cap(E_{\lambda_j}(k)\Delta E_{\lambda_j}(k-1)),$$

then there exists $C(n)>0$ such that

$$\begin{aligned}\Bigg|\int_{C_{\rho_j}^{\rho_j}(x,\nu)}(\chi_{E_{\lambda_j}(k)}-\chi_{E_{\lambda_j}(k-1)})dx-&\int_{C_{\rho_j}^{\rho_j}(x,\nu)\cap\partial E_{\lambda_j}(k)}\tilde d_{E_{\lambda_j}(k-1)}\,d\mathcal{H}^n\Bigg|\\ &\le C(n)w(1/\lambda_j)\Bigg|\int_{C_{\rho_j}^{\rho_j}(x,\nu)}|\chi_{E_{\lambda_j}(k)}-\chi_{E_{\lambda_j}(k-1)}|dx\Bigg|,\end{aligned}$$

where $\rho_j=\lambda_j^{-\sigma_1}/2,$ and

$$C_\rho^\rho(x,\nu):=\Big\{y\in\mathbb{R}^{n+1}:\ |(y-x)\cdot\nu|<\rho,\ \sqrt{|y-x|^2-|(y-x)\cdot\nu|^2}<\rho\Big\}.$$

Step 3. Now we estimate $R_k(j),\ \ k\ge2.$ Set $\sigma_2:=\frac{2n+5}{4(n+2)}.$ Let us define the following family

$$\mathsf{B}_k:=\{B(x),\ x\in\Omega^\varepsilon\cap(\partial E_{\lambda_j}(k-1)\cup\partial E_{\lambda_j}(k))\},$$

where

a) If $x\in\Omega^\varepsilon\cap\partial E_{\lambda_j}(k)$ has the property stated in Step 1, then $B(x):=C_\rho^\rho(x,\nu),$ where $\rho=\lambda_j^{\sigma_1}/2;$

b) otherwise $B(x):=B_{R\lambda_j^{-1/2}}(x).$

Notice that by Proposition 3.24 B_k is a cover of $\Omega^\varepsilon\cap(E_{\lambda_j}(k)\Delta E_{\lambda_j}(k-1)).$ Since $B_\rho(x)\subset C_\rho^\rho(x,\nu),$ we can use standard Besicovitch theorem (for balls) to find at most countable collection $\{B(x_i)\}\subset\mathsf{B}_k$ such that each point of $\Omega^\varepsilon\cap(E_{\lambda_j}(k)\Delta E_{\lambda_j}(k-1))$ is covered with at most $\mathfrak{b}(n)$ balls with twice radii.

Then in $B(x_i)$ satisfying a) we have

$$\begin{aligned}
&\left| \int_{B(x_i)} (\chi_{E_{\lambda_j}(k)} - \chi_{E_{\lambda_j}(k-1)})\phi\, dy - \int_{B(x_i)\cap \partial E_{\lambda_j}(k)} \tilde d_{E_{\lambda_j}(k-1)} \phi \, d\mathcal{H}^n \right| \\
&\quad \le |\phi(x_i)| \left| \int_{B(x_i)} (\chi_{E_{\lambda_j}(k)} - \chi_{E_{\lambda_j}(k-1)})\, dy - \int_{B(x_i)\cap \partial E_{\lambda_j}(k)} \tilde d_{E_{\lambda_k}(k-1)} \, d\mathcal{H}^n \right| \\
&\qquad + \left| \int_{B(x_i)} |\chi_{E_{\lambda_j}(k)} - \chi_{E_{\lambda_j}(k-1)}|\, |\phi - \phi(x_i)|\, dy + \int_{B(x_i)\cap \partial E_{\lambda_j}(k)} d_{E_{\lambda_k}(k-1)} |\phi - \phi(x_i)| \, d\mathcal{H}^n \right| \\
&\quad \le \Big(C(n) w(1/\lambda_j) \|\phi\|_\infty + \|\nabla \phi\|_\infty \lambda_j^{-\sigma_1} \Big) \int_{B(x_i)} |\chi_{E_{\lambda_j}(k)} - \chi_{E_{\lambda_j}(k-1)}|\, dy \\
&\qquad + \|\nabla \phi\|_\infty \lambda_j^{-\sigma_1} \int_{B(x_i)\cap \partial E_{\lambda_j}(k)} d_{E_{\lambda_j}(k-1)} \, d\mathcal{H}^n.
\end{aligned}$$

Now if we sum over balls satisfying a), we get

$$\begin{aligned}
A_k :=& \sum_{B(x_i)_{a)}} \int_{k/\lambda_j}^{(k+1)/\lambda_j} \lambda_j \left(\int_\Omega (\chi_{E_{\lambda_j}([\lambda_j t])} - \chi_{E_{\lambda_j}([\lambda_j t]-1)})\, \phi\, dx - \int_\Omega \tilde d_{E_{\lambda_j}([\lambda_j t]-1)}\, \phi \, d\mathcal{H}^n \right) dt \\
\le& \mathfrak{b}(n) \Big(C(n) w(1/\lambda_j) \|\phi\|_\infty + \|\nabla \phi\|_\infty \lambda_j^{-\sigma_1} \Big) |E_{\lambda_j}(k) \Delta E_{\lambda_j}(k-1)| \\
&+ \mathfrak{b}(n) \|\nabla \phi\|_\infty \lambda_j^{-\sigma_1-\sigma_2} P(E_{\lambda_j}(k)),
\end{aligned}$$

where in the last inequality we used $d_{E_{\lambda_j}(k-1)} \le \lambda_j^{-\sigma_2}$ in $B(x_i)$. In particular, by (3.74),

$$A_k \le \mathfrak{b}(n) \Big(C(n) w(1/\lambda_j) \|\phi\|_\infty + \|\nabla \phi\|_\infty \lambda_j^{-\sigma_1} \Big) |E_{\lambda_j}(k) \Delta E_{\lambda_j}(k-1)| + \mathfrak{b}(n) \|\nabla \phi\|_\infty \lambda_j^{-\sigma_1-\sigma_2} \frac{P(E_0)}{\kappa}.$$

Now take any ball $B(x_i)$ satisfying b), hence there exists $y \in B(x_i)$ such that $d_{E_{\lambda_j}(k-1)}(y) \ge \lambda_j^{-\sigma_2}$. Then clearly,

$$d_{E_{\lambda_j}(k-1)}(z) \ge \lambda_j^{-\sigma_2}/2, \quad \forall z \in B_{\lambda_j^{-\sigma_2}/2}(y).$$

Applying density estimates in Remark 3.23 to $B_{\lambda_j^{-\sigma_2}/2}(y) \cap (E_{\lambda_j}(k) \Delta E_{\lambda_j}(k-1))$ we establish

$$\int_{B_{\lambda_j^{-\sigma_2}/2}(y) \cap (E_{\lambda_j}(k) \Delta E_{\lambda_j}(k-1))} d_{E_{\lambda_j}(k-1)} \, dz \ge \left(\frac{\kappa}{2}\right)^{n+1} \omega_{n+1} \left(\frac{1}{2\lambda_j^{\sigma_2}} \right)^{n+2}.$$

Therefore, by choice of σ_2,

$$\begin{aligned}\int_{B(x_i)} |\chi_{E_{\lambda_j}(k)} - \chi_{E_{\lambda_j}(k-1)}|\,dx \le& \omega_{n+1}\left(R\lambda_j^{-1/2}\right)^{n+1}\\ &\le \left(\frac{4R}{\kappa}\right)^{n+1} 2\lambda_j^{3/4}\int_{B_{\lambda_j^{-\sigma_2}/2}(y)\cap(E_{\lambda_j}(k)\Delta E_{\lambda_j}(k-1))} d_{E_{\lambda_j}(k-1)}\,dz\\ &\le \left(\frac{4R}{\kappa}\right)^{n+1} 2\lambda_j^{3/4}\int_{B_{2R\lambda_j^{-1/2}}(x_i)\cap(E_{\lambda_j}(k)\Delta E_{\lambda_j}(k-1))} d_{E_{\lambda_j}(k-1)}\,dz.\end{aligned}$$

Now comparing $E_{\lambda_j}(k)$ to $E_{\lambda_j}(k)\setminus B_{R\lambda_j^{-1/2}}(x_i)$ we find

$$P(E_{\lambda_j}(k), B_{R\lambda_j^{-1/2}}(x_i)) \le C(n, \operatorname{diam}(E^+))(R\lambda_j^{-1/2})^n,$$

as a result (using also Proposition 3.24)

$$\int_{B_{R\lambda_j^{-1/2}}(x_i)\cap\partial E_{\lambda_j}(k)} d_{E_{\lambda_j}(k-1)}\,d\mathcal{H}^n \le C(n, \operatorname{diam}(E^+))(R\lambda_j^{-1/2})^{n+1}$$

$$\le \frac{C(n, \operatorname{diam}(E^+))}{\omega_{n+1}}\left(\frac{4R}{\kappa}\right)^{n+1} 2\lambda_j^{3/4}\int_{B_{2R\lambda_j^{-1/2}}(x_i)\cap(E_{\lambda_j}(k)\Delta E_{\lambda_j}(k-1))} d_{E_{\lambda_j}(k-1)}\,dz.$$

Then summing over the covering balls satisfying b), we obtain

$$\begin{aligned}B_k :=& \sum_{B(x_i)_{b)}}\int_{k/\lambda_j}^{(k+1)/\lambda_j}\lambda_j\left(\int_\Omega(\chi_{E_{\lambda_j}([\lambda_j t])} - \chi_{E_{\lambda_j}([\lambda_j t]-1)})\,\phi\,dx - \int_\Omega \tilde{d}_{E_{\lambda_j}([\lambda_j t]-1)}\,\phi\,d\mathcal{H}^n\right)dt\\ \le& \mathfrak{b}(n)\|\phi\|_\infty C(n,\kappa,\operatorname{diam}(E^+))\lambda_j^{-1/4}\left(\lambda_j\int_{E_{\lambda_j}(k)\Delta E_{\lambda_j}(k-1)} d_{E_{\lambda_j}(k-1)}\,dz\right)\\ \le& \mathfrak{b}(n)\|\phi\|_\infty C(n,\kappa,\operatorname{diam}(E^+))\lambda_j^{-1/4}\left(\mathcal{C}_\beta(E_{\lambda_j}(k-1),\Omega) - \mathcal{C}_\beta(E_{\lambda_j}(k),\Omega)\right).\end{aligned}$$

Since $R_k(j) \le A_k + B_k$, summation in $k \ge 2$ gives

$$\begin{aligned}\sum_{k=2}^N R_k(j) \le& \mathfrak{b}(n)\left(C(n)w(1/\lambda_j)\|\phi\|_\infty + \|\nabla\phi\|_\infty\lambda_j^{-\sigma_1}\right)\sum_{k=2}^N |E_{\lambda_j}(k)\Delta E_{\lambda_j}(k-1)|\\ &+ \mathfrak{b}(n)\|\nabla\phi\|_\infty\lambda_j^{1-\sigma_1-\sigma_2}\frac{P(E_0)}{\kappa}\frac{N-1}{\lambda_j}\\ &+ \mathfrak{b}(n)\|\phi\|_\infty C(n,\kappa,\operatorname{diam}(E^+))\lambda_j^{-1/4}\sum_{k=2}^N\left(\mathcal{C}_\beta(E_{\lambda_j}(k-1),\Omega) - \mathcal{C}_\beta(E_{\lambda_j}(k),\Omega)\right)\end{aligned}$$

Now using Proposition 3.51, the relation $N \le T\lambda_j$ and (3.74), we get

$$\sum_{k=2}^{N} R_k(j) \le \mathfrak{b}(n)\Big(C(n)w(1/\lambda_j)\|\phi\|_\infty + \|\nabla\phi\|_\infty \lambda_j^{-\sigma_1}\Big)\left(\frac{2^{n+1}\omega_{n+1}\gamma(n,\kappa)\mathfrak{b}(n)}{\kappa c(n,\kappa)}\,P(E_0)\,T\right)$$
$$+\,\mathfrak{b}(n)\|\nabla\phi\|_\infty \lambda_j^{1-\sigma_1-\sigma_2}\,\frac{P(E_0)}{\kappa}\,T$$
$$+\,\mathfrak{b}(n)\|\phi\|_\infty C(n,\kappa,\mathrm{diam}(E^+))\lambda_j^{-1/4}P(E_0),$$

Notice that $\sigma_1+\sigma_2>1$. Since $R_1(j)\to 0$ as $j\to+\infty$ by Lemma 3.17, the estimate for $\sum_{k=2}^{N} R_k(j)$ yields (3.97). $\square$

Proof of Theorem 3.47. Lemma 3.50, (3.86) and [68, Theorem 4.4.2] imply that there exist a (not relabelled) subsequence and a function $v:[0,+\infty)\times\Omega\to\mathbb{R}$ satisfying (3.87)-(3.89). In particular, from (3.87) it follows that $H_{E(t)} := v(t,\cdot)\big|_{\Omega\cap\partial^*E(t)} \in L^2(\Omega\cap\partial^*E(t),\mathcal{H}^n\llcorner(\Omega\cap\partial^*E(t)))$ for a.e. $t>0$. Let us prove that $H_{E(t)}$ is the distributional mean curvature of $E(t)$ for a.e. $t\ge 0$. Fixing $t\ge 0$, by the divergence formula (1.3) for any $\phi\in C_c^1(\mathbb{R}^{n+1},\mathbb{R}^{n+1})$ one has

$$\int_{E_{\lambda_j}([\lambda_j t])} \mathrm{div}\,\phi dx - \int_{\Omega\cap\partial^*E_{\lambda_j}([\lambda_j t])} \phi\cdot\nu_{E_{\lambda_j}([\lambda_j t])}\,d\mathcal{H}^n = \int_{\partial\Omega\cap\partial^*E_{\lambda_j}([\lambda_j t])} \phi_{n+1} d\mathcal{H}^n.$$

Hence, from (3.69) and (3.71) we get

$$\int_{E(t)} \mathrm{div}\,\phi dx - \int_{\Omega\cap\partial^*E(t)} \phi\cdot\nu_{E(t)}\,d\mathcal{H}^n = \lim_{j\to+\infty}\int_{\mathrm{Tr}(E_{\lambda_j}([\lambda_j t]))} \phi_{n+1} d\mathcal{H}^n. \tag{3.100}$$

The left-hand-side of (3.100) is $\int_{\mathrm{Tr}(E(t))}\phi_{n+1}d\mathcal{H}^n,$ therefore,

$$\mathcal{H}^n\llcorner\mathrm{Tr}(E_{\lambda_j}([\lambda_j t])) \overset{w^*}{\rightharpoonup} \mathcal{H}^n\llcorner\mathrm{Tr}(E(t)) \qquad \text{as } j\to+\infty. \tag{3.101}$$

Combining this with (3.86) we get

$$\mathcal{H}^n\llcorner\partial^*E_{\lambda_j}([\lambda_j t]) \overset{w^*}{\rightharpoonup} \mathcal{H}^n\llcorner\partial^*E(t) \quad \text{as } j\to+\infty \text{ for a.e. } t\ge 0.$$

Take $\eta\in C_c^1([0,+\infty))$ and an admissible $X\in C_c^1(\overline{\Omega},\mathbb{R}^{n+1})$. By (3.86) and [91, formula (4.2)] for a.e. $t\ge 0$ and for every $F\in C_c(\mathbb{R}^{n+1}\times\mathbb{R}^{n+1})$ one has

$$\lim_{j\to+\infty}\int_{\Omega\cap\partial^*E_{\lambda_j}([\lambda_j t])} F(x,\nu_{E_{\lambda_j}([\lambda_j t])}(x))\,d\mathcal{H}^n = \int_{\Omega\cap\partial^*E(t)} F(x,\nu_{E(t)}(x))\,d\mathcal{H}^n. \tag{3.102}$$

In particular, taking $F\in C_c(\overline{\Omega}\times\mathbb{R}^{n+1})$ such that $F(x,\xi)=\mathrm{div}\,X(x)-\xi\cdot\nabla X(x)\xi$ in $\Omega\times\{|\xi|\le 2\}$, by the dominated convergence theorem, (3.85) and (3.89), for $\Psi(t,x)=\eta(t)X(x)$ we establish

$$\int_0^{+\infty}\eta(t)\int_{\Omega\cap\partial^*E(t)}F(x,\nu_{E(t)}(x))d\mathcal{H}^n dt=\lim_{j\to+\infty}\int_0^{+\infty}\int_{\Omega\cap\partial^*E_{\lambda_j}([\lambda_j t])}\eta(t)F(x,\nu_{E_{\lambda_j}([\lambda_j t])})d\mathcal{H}^n dt$$

$$=\lim_{j\to+\infty}\int_0^{+\infty}\int_{\Omega\cap\partial^*E_{\lambda_j}([\lambda_j t])}v_{\lambda_j}\nu_{E_{\lambda_j}([\lambda_j t])}\cdot\Psi(t,x)d\mathcal{H}^n dt$$

$$=\int_0^{+\infty}\int_{\Omega\cap\partial^*E(t)}v\nu_{E(t)}\cdot\Psi(t,x)d\mathcal{H}^n dt=\int_0^{+\infty}\eta(t)\int_{\Omega\cap\partial^*E(t)}H_{E(t)}\nu_{E(t)}\cdot X\,d\mathcal{H}^n dt.$$

Since $\eta\in C_c^1([0,+\infty))$ is arbitrary, for a.e. $t\geq 0$ we get

$$\int_{\Omega\cap\partial^*E(t)}(\operatorname{div}X-\nu_{E(t)}\cdot(\nabla X)\nu_{E(t)})d\mathcal{H}^n=\int_{\Omega\cap\partial^*E(t)}H_{E(t)}\nu_{E(t)}\cdot X\,d\mathcal{H}^n,$$

hence $H_{E(t)}$ is the generalized mean curvature of $\Omega\cap\partial^*E(t)$.

Let us show (3.90). Take $\phi\in C_c^1([0,+\infty)\times\Omega)$. By a change of variables we have

$$\int_{1/\lambda_j}^{+\infty}\Big[\int_{E_{\lambda_j}([\lambda_j t])}\phi dx-\int_{E_{\lambda_j}([\lambda_j t]-1)}\phi dx\Big]dt$$

$$=\int_{1/\lambda_j}^{+\infty}\int_{E_{\lambda_j}([\lambda_j t])}(\phi(t,x)-\phi(t+1/\lambda_j,x))dxdt-\frac{1}{\lambda_j}\int_{E(0)}\phi(x,0)\,dx.$$

Since $E(0)=E_0$, from (3.81) we get

$$\lim_{j\to+\infty}\int_{1/\lambda_j}^{+\infty}\lambda_j\Big[\int_{E_{\lambda_j}([\lambda_j t])}\phi dx-\int_{E_{\lambda_j}([\lambda_j t]-1)}\phi dx\Big]dt=-\int_0^{+\infty}\int_{E(t)}\frac{\partial\phi}{\partial t}(t,x)\,dxdt-\int_{E_0}\phi(x,0)dx.$$

Therefore, (3.97), (3.88) and the definition of $H_{E(t)}$ imply

$$\int_0^{+\infty}\int_{E(t)}\partial_t\phi\,dxdt+\int_{E_0}\phi(x,0)dx=\lim_{j\to+\infty}\int_0^{+\infty}\int_{\Omega\cap\partial E_{\lambda_j}([\lambda_j t])}v_{\lambda_j}\phi\,d\mathcal{H}^n dt$$

$$=\int_0^{+\infty}\int_{\Omega\cap\partial^*E(t)}H_{E(t)}\phi\,d\mathcal{H}^n dt.$$

(ii) Take an admissible $X\in C_c^1(\overline{\Omega},\mathbb{R}^{n+1})$ and $\eta\in C_c^1([0,+\infty))$. From (3.84)

$$\begin{aligned}&\int_0^{+\infty}\eta(t)\int_{\Omega\cap\partial^*E_{\lambda_j}([\lambda_j t])}\Big(\operatorname{div}X-\nu_{E_{\lambda_j}([\lambda_j t])}\cdot(\nabla X)\nu_{E_{\lambda_j}([\lambda_j t])}\Big)d\mathcal{H}^n dt\\&\quad-\int_0^{+\infty}\eta(t)\int_{\Omega\cap\partial^*E_{\lambda_j}([\lambda_j t])}v_{\lambda_j}X\cdot\nu_{E_{\lambda_j}([\lambda_j t])}\,d\mathcal{H}^n dt\\&=\int_0^{+\infty}\eta(t)\int_{\partial^*\mathrm{Tr}(E_{\lambda_j}([\lambda_j t]))}\beta X'\cdot\nu'_{\mathrm{Tr}(E_{\lambda_j}([\lambda_j t]))}\,d\mathcal{H}^{n-1}.\end{aligned}\tag{3.103}$$

Let $\{\lambda_{j_l}\}_{l\geq 1}$ be any subsequence of $\{\lambda_j\}$. By the uniform bound (3.91) on the perimeters and by compactness there exists a further subsequence $\{\lambda_{j_{l_k}}\}_{k\geq 1}$ of $\{\lambda_{j_l}\}_{l\geq 1}$ and a set $\hat F \in BV(\mathbb{R}^n,\{0,1\})$ such that $\mathrm{Tr}(E_{j_{l_k}}([j_{l_k}t]))\to \hat F$ in $L^1(\mathbb{R}^n)$ and[4]

$$\nu'_{\mathrm{Tr}(E_{\lambda_{j_{l_k}}}([\lambda_{j_{l_k}}t]))}\,\mathcal{H}^{n-1}\llcorner\partial^*\mathrm{Tr}(E_{\lambda_{j_{l_k}}}([\lambda_{j_{l_k}}t]))\overset{w^*}{\rightharpoonup}\nu'_{\hat F}\,\mathcal{H}^{n-1}\llcorner\partial^*\hat F\quad\text{as } k\to+\infty$$

for a.e. $t\geq 0$. By (3.101) for every $\phi\in C_c(\mathbb{R}^n)$ we have

$$\int_{\mathrm{Tr}(E(t))}\phi\,d\mathcal{H}^n=\lim_{k\to+\infty}\int_{\mathrm{Tr}(E_{\lambda_{j_{l_k}}}([\lambda_{j_{l_k}}t]))}\phi\,d\mathcal{H}^n=\int_{\hat F}\phi\,d\mathcal{H}^n.$$

Whence, $\hat F=\mathrm{Tr}(E(t))$. Therefore

$$\nu'_{\mathrm{Tr}(E_{\lambda_j}([\lambda_j t]))}\,\mathcal{H}^{n-1}\llcorner\partial^*\mathrm{Tr}(E_{\lambda_j}([\lambda_j t]))\overset{w^*}{\rightharpoonup}\nu'_{\mathrm{Tr}(E(t))}\,\mathcal{H}^{n-1}\llcorner\partial^*\mathrm{Tr}E(t)\quad\text{as } j\to+\infty.$$

Now taking limit in (3.103), using (3.102),(3.89) and applying the dominated convergence theorem on the right-hand-side we get (3.92). □

3.8 Local well-posedness

In this appendix we sketch the proof of short time existence and uniqueness of smooth hypersurfaces moving with normal velocity equal to their mean curvature in Ω and meeting the boundary $\partial\Omega$ at a prescribed (not necessarily constant) angle. The following theorem is a generalization of [72, Theorem 1], where short time existence and uniqueness have been proven for constant β.

Theorem 3.53 (Short time existence and uniqueness). *Let $\beta\in C^{1+\alpha}(\partial\Omega)$, $\|\beta\|_\infty\leq 1-2\kappa$, $\kappa\in(0,\frac12]$ and $E_0\subset\Omega$ be a bounded open set such that $\Gamma_0=\overline{\Omega\cap\partial E_0}$ is a bounded $C^{3+\alpha}$-hypersurface, $\alpha\in(0,1)$. Assume that $\mathcal{U}\subset\mathbb{R}^n$ is a bounded open set with $C^{3+\alpha}$-boundary, $p^0\in C^{3+\alpha}(\overline{\mathcal{U}},\mathbb{R}^{n+1})$ is a parametrization of Γ_0 such that $p^0_{n+1}>0$ in $\mathcal{U}$, $p^0_{n+1}=0$ on $\partial\mathcal{U}$, and*

$$-e_{n+1}+\beta(p^0)\nu_0=Dp^0[n^0]\quad\text{on } \partial\mathcal{U},\tag{3.104}$$

where $n^0=(n^0_1,\dots,n^0_n)$ is the outward unit normal to $\partial\mathcal{U}$, $\nu_0=\nu(p^0)$ is the outward unit normal of Γ_0 at p^0 and $Dp^0[n^0]=\sum_{j=1}^n n^0_j p^0_{\sigma_j}$. Then there exists $T_0=T_0(\|\beta\|_{C^{1+\alpha}},\|p^0\|_{C^{3+\alpha}})>0$, a unique family of bounded open sets $\{E(t)\subset\Omega:\ t\in[0,T_0]\}$ with a parametrization $p\in C^{1+\alpha/2,2+\alpha}([0,T_0]\times\overline{\mathcal{U}},\mathbb{R}^{n+1})$ of $\Gamma(t)=\overline{\Omega\cap\partial E(t)}$ solving the parabolic system

$$p_t=\mathrm{trace}((Dp\cdot(Dp)^T)^{-1}D^2p)\ \ \text{in } (0,T_0)\times\mathcal{U},\tag{3.105}$$

where $(Dp\cdot(Dp)^T)_{ij}=p_{\sigma_i}\cdot p_{\sigma_j}$ and $(D^2p)_{ij}=p_{\sigma_i\sigma_j}$, coupled with the initial condition $p(0,\cdot)=p^0$, the boundary conditions

$$\begin{cases}p_{n+1}(t,\cdot)=0 & \text{on } \partial\mathcal{U} \text{ for any } t\in[0,T_0],\\ e_{n+1}\cdot\nu(p(t,\cdot))=\beta(p(t,\cdot)) & \text{on } \partial\mathcal{U} \text{ for any } t\in[0,T_0],\end{cases}$$

[4] Arguing, for example, as in (3.83).

and the orthogonality conditions

$$Dp^0[n^0]\cdot\tau_{0i} = 0 \text{ on } [0,T_0]\times\partial\mathcal{U} \text{ for every } i=1,\dots,n-1, \tag{3.106}$$

where $\nu(p(t,\cdot))$ *is the outward unit normal to* $\Gamma(t)$ *at* $p(t,\cdot)$ *and* $\tau_{01},\dots,\tau_{0n-1}\in\mathbb{R}^n\times\{0\}$ *is a basis for the tangent space of* $\Gamma_0\cap\partial\Omega$ *at* p^0.

Remark 3.54. Assumption (3.104) on p^0 is not restrictive. Indeed, if $q:\partial\mathcal{U}\to\Gamma_0\cap\partial\Omega$ is a $C^{3+\alpha}$ parametrization of the contact set, we may extend it to a sufficiently small tubular neighborhood $S:=\{x\in\mathcal{U}:\ \mathrm{dist}(x,\partial\mathcal{U})<\varepsilon\}$ of $\partial\mathcal{U}$ in $\mathcal{U}$ with the properties that q is a $C^{3+\alpha}$ diffeomorphism, $q(S)\subset\Gamma_0$ and

$$q(\sigma) = q(\varsigma) + |\sigma-\varsigma|(e_{n+1} - \beta(q(\varsigma))\nu_0(q(\varsigma))) + O(|\sigma-\varsigma|^2),$$

where ς is the projection of $\sigma\in S$ on $\partial\mathcal{U}$. Since $\sigma=\varsigma-|\sigma-\varsigma|n^0(\varsigma)$, it follows

$$\nabla q(\varsigma)\, n^0(\varsigma) = -e_{n+1} + \beta(q(\varsigma))\nu_0(q(\varsigma)),$$

which is (3.104). Now we may arbitrarily extend q to a $C^{3+\alpha}$ diffeomorphism in $\overline{\mathcal{U}}$ such that $q(\overline{\mathcal{U}})=\Gamma_0$.

Remark 3.55. The unit normal to $\Gamma(t)$ at the point $p(t,\sigma_1,\dots,\sigma_n)\in\Gamma(t)$ can be written with a (standard) abuse of notation $\nu=\nu(p(t,\sigma_1,\dots,\sigma_n))=\frac{\tilde\nu}{|\tilde\nu|}$, where

$$\tilde\nu := \tilde\nu(p_\sigma) = \det\begin{bmatrix} e_1 & e_2 & \dots & e_n & e_{n+1} \\ & & p_{\sigma_1} & & \\ & & p_{\sigma_2} & & \\ & & \vdots & & \\ & & p_{\sigma_n} & & \end{bmatrix}.$$

Proof of Theorem 3.53. The idea of the proof is standard: first we linearize the equation around the boundary conditions, then prove the existence result for the linearized system and finally we use a fixed point argument.

Step 1. Let us linearize system (3.105) fixing some $t_0>0$. Let $X(t_0)\subset C^{1+\alpha/2,2+\alpha}([0,t_0]\times\overline{\mathcal{U}},\mathbb{R}^{n+1})$ be the nonempty convex set consisting of all functions $w\in C^{1+\alpha/2,2+\alpha}([0,t_0]\times\overline{\mathcal{U}},\mathbb{R}^{n+1})$ such that

1) $w(0,\cdot)=p^0$,

2) $w_{n+1}(t,\cdot)=0$ on $\partial\mathcal{U}$ for any $t\in[0,t_0]$,

3) $\sum_{j=1}^n n_j^0 w_{\sigma_j}\cdot\tau_{0i}=0$ on $[0,t_0]\times\partial\mathcal{U}$ for every $i=1,\dots,n-1$.

For $w\in X(t_0)$ set $f(t,w):=\mathrm{trace}\big[\big((Dw\cdot(Dw)^T)^{-1}-(Dp^0\cdot(Dp^0)^T)^{-1}\big)D^2w\big]$. Then (3.105) is equivalent to

$$w_t = \mathrm{trace}\big[(Dp^0\cdot(Dp^0)^T)^{-1}D^2w\big] + f(t,w).$$

Notice that

$$|f(t,w)| \le c(\|p^0\|_{C^1(\overline{\mathcal{U}})})\|w\|_{C^{0,2}([0,t_0]\times\overline{\mathcal{U}})}\|w-p_0\|_{C^{0,1}([0,t_0]\times\overline{\mathcal{U}})},$$

where $c(\|p^0\|_{C^1(\overline{\mathcal{U}})}) > 0$. Now we linearize the contact angle condition. Since we have $e_{n+1}\cdot\nu(p^0) = \beta(p^0)$, from Remark 3.55 it follows that

$$e_{n+1}\cdot\big(\tilde{\nu}(w_\sigma)-\tilde{\nu}(p^0_\sigma)\big) = \beta(w)|\tilde{\nu}(w_\sigma)| - \beta(p^0)|\tilde{\nu}(p^0_\sigma)|. \tag{3.107}$$

Let $H_1(t,w) := \tilde{\nu}(w_\sigma)-\tilde{\nu}(p^0_\sigma) - D\tilde{\nu}(p^0_\sigma)[w_\sigma - p^0_\sigma]$, where

$$D\tilde{\nu} = \begin{bmatrix} D_{p_{\sigma_1}}\tilde{\nu}^1 & D_{p_{\sigma_2}}\tilde{\nu}^1 & \dots & D_{p_{\sigma_n}}\tilde{\nu}^1 \\ D_{p_{\sigma_1}}\tilde{\nu}^2 & D_{p_{\sigma_2}}\tilde{\nu}^2 & \dots & D_{p_{\sigma_n}}\tilde{\nu}^2 \\ \vdots & \vdots & \dots & \vdots \\ D_{p_{\sigma_1}}\tilde{\nu}^{n+1} & D_{p_{\sigma_2}}\tilde{\nu}^{n+1} & \dots & D_{p_{\sigma_n}}\tilde{\nu}^{n+1} \end{bmatrix}, \qquad q_\sigma = \begin{bmatrix} q_{\sigma_1} \\ q_{\sigma_2} \\ \vdots \\ q_{\sigma_n} \end{bmatrix} = \begin{bmatrix} (q_1)_{\sigma_1} & \dots & (q_{n+1})_{\sigma_1} \\ (q_1)_{\sigma_2} & \dots & (q_{n+1})_{\sigma_2} \\ \dots & \vdots & \dots \\ (q_1)_{\sigma_n} & \dots & (q_{n+1})_{\sigma_n} \end{bmatrix}$$

and

$$D\tilde{\nu}[q_\sigma] = \begin{bmatrix} \sum_{i=1}^n D_{p_{\sigma_i}}\tilde{\nu}^1\cdot q_{\sigma_i} \\ \sum_{i=1}^n D_{p_{\sigma_i}}\tilde{\nu}^2\cdot q_{\sigma_i} \\ \vdots \\ \sum_{i=1}^n D_{p_{\sigma_i}}\tilde{\nu}^{n+1}\cdot q_{\sigma_i} \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^n\sum_{j=1}^{n+1} D_{(p_j)_{\sigma_i}}\tilde{\nu}^1\cdot (q_j)_{\sigma_i} \\ \sum_{i=1}^n\sum_{j=1}^{n+1} D_{(p_j)_{\sigma_i}}\tilde{\nu}^2\cdot (q_j)_{\sigma_i} \\ \vdots \\ \sum_{i=1}^n\sum_{j=1}^{n+1} D_{(p_j)_{\sigma_i}}\tilde{\nu}^{n+1}\cdot (q_j)_{\sigma_i} \end{bmatrix}$$

Clearly, $|H_1(t,w)| = O\Big(\|w-p^0\|^2_{C^{0,1}([0,t_0]\times\overline{\mathcal{U}})}\Big)$. Moreover,

$$|\tilde{\nu}(w_\sigma)| = |\tilde{\nu}(p^0_\sigma)| + \nu(p^0)\cdot D\tilde{\nu}(p^0_\sigma)[w_\sigma - p^0_\sigma] + H_2(t,w)$$

with $|H_2(t,w)| = O\Big(\|w-p^0\|^2_{C^{0,1}([0,t_0]\times\overline{\mathcal{U}})}\Big)$. Finally, since $\beta\in C^{1+\alpha}(\partial\Omega)$ we have

$$\beta(w)|\tilde{\nu}(w_\sigma)| - \beta(p^0)|\tilde{\nu}(p^0_\sigma)| = \beta(p^0)\nu(p^0)\cdot D\tilde{\nu}(p^0_\sigma)[w_\sigma - p^0_\sigma] + H_3(t,w),$$

where $H_3(t,w) = O\Big(\|w-p^0\|^2_{C^{0,1}([0,t_0]\times\overline{\mathcal{U}})}\Big)$. Thus, (3.107) is equivalent to

$$(e_{n+1} - \beta(p^0)\nu(p^0))\cdot D\tilde{\nu}(p^0_\sigma)[w_\sigma] = (e_{n+1} - \beta(p^0)\nu(p^0))\cdot D\tilde{\nu}(p^0_\sigma)[p^0_\sigma] + H_4(t,w),$$

where $H_4(t,w) = O\Big(\|w-p^0\|^2_{C^{0,1}([0,t_0]\times\overline{\mathcal{U}})}\Big)$.

Thus we have the following linear parabolic system of equations

$$\mathcal{L}(\sigma,\partial_t,\partial_\sigma)w = f \text{ in } (0,t_0)\times\mathcal{U}$$

subject to the boundary conditions $\mathcal{B}_\beta(\varsigma,\partial_\sigma)w = F(t,\varsigma)$ on $[0,t_0]\times\partial\mathcal{U}$, where

$$F(t,\varsigma) = \Big[0,\ (e_{n+1} - \beta(p^0)\nu(p^0))\cdot D\tilde{\nu}(p^0_\sigma)[p^0_\sigma] + H_4(t,w),\ \underbrace{0,\ \dots\ ,0}_{(n-1)-\text{times}}\Big]^T$$

and, under the notation $\{g_0\}^{ij} = \{p^0_{\sigma_i} \cdot p^0_{\sigma_j}\}^{-1}$, $\tilde\nu_0 = \tilde\nu(p^0_\sigma)$, $\beta_0 = \beta(p^0)$ the $(n+1)\times(n+1)$-matrices $\mathcal{L}(\sigma, t, \xi, \zeta)$ and $\mathcal{B}_\beta(\varsigma, \xi)$, $\xi \in \mathbb{R}^n$, $\zeta \in \mathbb{C}$ are defined as follows:

$$\mathcal{L}(\sigma, \zeta, \xi) := \mathrm{diag}\left(\zeta - \sum_{i,j=1}^n g_0^{ij}\xi_i\xi_j, \zeta - \sum_{i,j=1}^n g_0^{ij}\xi_i\xi_j, \dots, \zeta - \sum_{i,j=1}^n g_0^{ij}\xi_i\xi_j\right),$$

$$\mathcal{B}_\beta(\varsigma, \xi) := \begin{bmatrix} 0 & \dots & 1 \\ \sum\limits_{k=1}^{n+1}\sum\limits_{i=1}^n (-\delta_{k,n+1} - \beta_0\nu_0^k) D_{(p_1)_{\sigma_i}}\tilde\nu_0^k\xi_i & \dots & \sum\limits_{k=1}^{n+1}\sum\limits_{i=1}^n (-\delta_{k,n+1} - \beta_0\nu_0^k) D_{(p_{n+1})_{\sigma_i}}\tilde\nu_0^k\xi_i \\ \tau_{01}^1 \sum\limits_{i=1}^n n_i^0\xi_i & \dots & \tau_{01}^{n+1}\sum\limits_{i=1}^n n_i^0\xi_i \\ \vdots & \vdots & \vdots \\ \tau_{0n-1}^1 \sum\limits_{i=1}^n n_i^0\xi_i & \dots & \tau_{0n-1}^{n+1}\sum\limits_{i=1}^n n_i^0\xi_i \end{bmatrix},$$

where the first row must be intended as $[0, \dots, 0, 1]$.

Step 2. Now we check the compatibility conditions [103]. Take any $\varsigma \in \partial\mathcal{U}$ and let θ be in the tangent space of $\partial\mathcal{U}$ at ς. Let $\lambda_0 := \lambda_0(\varsigma, \zeta, \theta)$ be a solution of the quadratic equation

$$h(\lambda; \varsigma, \zeta, \theta) := \zeta + \sum_{i,j=1}^n g_0^{ij}\theta_i\theta_j - 2\lambda\sum_{i,j=1}^n g_0^{ij}\theta_i n_j^0 + \lambda^2\sum_{i,j=1}^n g_0^{ij} n_i^0 n_j^0 = 0$$

in $\lambda \in \mathbb{C}$ with positive imaginary part. Notice that $\det\mathcal{L} = (h(\lambda; \varsigma, \zeta, \theta))^{n+1}$ and

$$\hat{\mathcal{L}} = (\det\mathcal{L})\mathcal{L}^{-1} = \mathrm{diag}((h(\lambda; \varsigma, \zeta, \theta))^n, \dots, (h(\lambda; \varsigma, \zeta, \theta))^n).$$

In order to prove the compatibility conditions we should prove that the rows of matrix

$$\mathcal{B}_\beta(\varsigma, i(\theta - \lambda n^0))\hat{\mathcal{L}}(x, \zeta, i(\theta - \lambda n^0))$$

are linearly independent modulo the polynomial $(\lambda - \lambda_0)^{n+1}$ whenever $\Re(\zeta) \ge 0$, $|\zeta| > 0$. According to the definitions of $\mathcal{L}$ and $\mathcal{B}_\beta$ one checks [72] that the compatibility conditions are equivalent to the conditions

$$c_1 e_{n+1} + c_2\tilde\nu(p^0) + \sum_{i=1}^{n-1} c_{i+2}\tau_{0i} = 0 \iff c_1 = c_2 = \dots = c_{n+1} = 0.$$

Since a basis of the tangent space $\{\tau_{0i}\}_{i=1}^{n-1}$ of $\Gamma_0 \cap \partial\Omega$ belongs to the horizontal subspace of $\mathbb{R}^{n+1}$ and $\tilde\nu(p^0)$ is normal to $\Gamma_0 \cap \partial\Omega$ at p^0 we have $c_3 = \dots = c_{n+1} = 0$. Moreover, since $|\beta| \le 1 - 2\kappa$, and Γ_0 satisfies the contact angle condition, e_{n+1} and $\tilde\nu(p^0)$ are linearly independent, i.e. $c_1 = c_2 = 0$.

Step 3. By [103, Theorem 4.9] since $\partial\mathcal{U} \in C^{3+\alpha}$, $\beta \in C^{1+\alpha}(\partial\Omega)$ and the compatibility conditions hold, for any $\tilde f, \tilde F \in C^{0,\alpha}([0, t_0] \times \overline{\mathcal{U}})$, $p^0 \in C^{3+\alpha}(\overline{\mathcal{U}})$ there exists a unique solution

$w \in C^{1+\alpha/2,2+\alpha}([0,t_0]\times\overline{\mathcal{U}})$ such that

$$w_t = \mathrm{tr}((Dp^0\cdot(Dp^0)^t)^{-1}D^2w)+\tilde{f},$$

$$w(0,\cdot)=p^0,$$

$$w_{n+1}(t,\cdot)=0 \qquad \text{on } \partial\mathcal{U} \text{ for any } t\in[0,t_0],$$

$$(e_{n+1}-\beta(p^0)\nu(p^0))\cdot D\tilde{\nu}(p^0)[w_\sigma]=(e_{n+1}-\beta(p^0)\nu(p^0))\cdot D\tilde{\nu}(p^0)[p^0_\sigma]+\tilde{F}(t,x) \quad \text{on } [0,t_0]\times\partial\mathcal{U},$$

$$\left(\sum_{j=1}^n n_j^0 w_{\sigma_j}\right)\cdot\tau_{0i}=0 \qquad \text{on } [0,t_0]\times\partial\mathcal{U} \text{ and } i=1,\dots,n-1.$$

Step 4. Finally, mimicking [54] we can prove the existence of and uniqueness of solution (3.105)-(3.106) in time interval $[0,T_0]$ for some sufficiently small $T_0>0$ depending on $\|\beta\|_{C^{1+\alpha}}$ and $\|p^0\|_{C^{3+\alpha}}$. □

We call $E(t)$ the smooth flow starting from E_0.

Proposition 3.56 (Comparison for strong solutions). *Let $\beta_i\in(-1,1)$, $E_0^{(i)}\subset\Omega$ be bounded sets such that $\overline{\Omega\cap\partial E_0^{(i)}}$ are $C^{3+\alpha}$ hypersurfaces, and the smooth flows $E^{(i)}(t)$ starting from $E_0^{(i)}$ exist in $[0,T_0]$, $i=1,2$. If $\beta_1\le\beta_2$ and $\mathrm{dist}(\Omega\cap\partial E_0^{(1)},\Omega\cap\partial E_0^{(2)})>0$, then $\mathrm{dist}(\Omega\cap\partial E^{(1)}(t),\Omega\cap\partial E^{(2)}(t))>0$ for all $t\in[0,T_0]$.*

Proof. The proof is an adaptation of the classical one (see for instance [16]). □

Chapter 4

Minimizing movements for partitions

The present chapter is a joint work with G. Belletini [20] and devoted to prove the existence of a weak global in time anisotropic forced mean curvature flow of a bounded partition using the method of minimizing movements. We also show that the Euclidean minimizing movement solution starting from a partition made by a union of bounded convex sets at a positive distance agrees with the classical mean curvature flow, and the motion is stable with respect to the Hausdorff convergence of the initial partition.

The plan of the chapter is the following. Section 4.1 is devoted to the definitions of partitions and density estimates for almost-minimizers of the anisotropic perimeter. In Section 4.2 we prove the existence of minimizers of Almgren-Taylor-Wang-type functional F_H^Φ in $\mathbb{P}_b(N+1)$ (Theorem 4.10), the density estimates (Theorem 4.13), and – one of our main results – the existence of GMM for F_H^Φ (Theorem 4.16). Finally in Section 4.3 we prove that the minimizers of $F_H^\Phi(\cdot,\mathcal{G};\lambda)$ with disjoint $\mathcal{G}$ (Definition 4.18) is also disjoint provided λ is large enough (Theorem 4.22) and as a nontrivial application of this fact, we obtain consistency and stability results for convex and disjoint partitions (Theorem 4.19 and Theorem 4.24).

4.1 Partitions

Definition 4.1 (Partition). *Given an integer $N \geq 2$, an N-tuple $\mathcal{C} = (C_1, \dots, C_N)$ of subsets of $\mathbb{R}^n$ is called an N* **-partition** *of $\mathbb{R}^n$ (a partition, for short) if*

(a) $C_i \in BV_{\mathrm{loc}}(\mathbb{R}^n, \{0,1\})$ *for every* $i = 1, \dots, N$,

(b) $\sum\limits_{i=1}^{N} |C_i \cap K| = |K|$ *for each compact* $K \subseteq \mathbb{R}^n$.

The collection of all N-partitions of $\mathbb{R}^n$ is denoted by $\mathbb{P}(N)$. Our assumptions $C_i = C_i^{(1)}$ implies $C_i \cap C_j = \emptyset$ for $i \neq j$. Notice also that we do not exclude the case $C_i = \emptyset$.

The elements of $\mathbb{P}(N)$ are denoted by calligraphic letters $\mathcal{A}, \mathcal{B}, \mathcal{C}, \dots$ and the entries (also called components) of $\mathcal{A} \in \mathbb{P}(N)$ by the corresponding roman letters $(A_1, \dots, A_N)$. The functional

$$(\mathcal{A}, \Omega) \in \mathbb{P}(N) \times \mathrm{Op}(\mathbb{R}^n) \mapsto \mathrm{Per}\,(\mathcal{A}, \Omega) := \sum_{j=1}^{N} P(A_j, \Omega)$$

is called the perimeter of the partition $\mathcal{A}$ in Ω. For simplicity, we write $\mathrm{Per}\,(\mathcal{A}) := \mathrm{Per}\,(\mathcal{A}, \mathbb{R}^n)$. More generally, given an N-tuple $\Phi := (\phi_1, \dots, \phi_N)$ of norms and $\Omega \in \mathrm{Op}(\mathbb{R}^n)$ we define the Φ-perimeter of $\mathcal{A} \in \mathbb{P}(N)$ in Ω as

$$\mathrm{Per}_{\Phi}(\mathcal{A}, \Omega) := \sum_{j=1}^{N} P_{\phi_j}(A_j, \Omega).$$

In what follows we suppose that there exists $0 < \kappa_1 \le \kappa_2$ such that

$$\kappa_1 |\xi| \le \phi_j(\xi) \le \kappa_2 |\xi| \qquad \forall \xi \in \mathbb{R}^n, \quad \forall j. \tag{4.1}$$

We set

$$\mathcal{A} \Delta \mathcal{B} := \bigcup_{j=1}^{N} A_j \Delta B_j$$

and

$$|\mathcal{A} \Delta \mathcal{B}| := \sum_{j=1}^{N} |A_j \Delta B_j|,$$

where Δ is the symmetric difference of sets, i.e. $E \Delta F = (E \setminus F) \cup (F \setminus E)$.

We say that the sequence $\{\mathcal{A}^{(k)}\} \subseteq \mathbb{P}(N)$ *converges* to $\mathcal{A} \in \mathbb{P}(N)$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ if

$$|(\mathcal{A}^{(k)} \Delta \mathcal{A}) \cap K| := \sum_{j=1}^{N} |(A_j^{(k)} \Delta A_j) \cap K| \to 0 \qquad \text{as } k \to +\infty$$

for every compact set $K \subseteq \mathbb{R}^n$. Since $E \in BV_{\mathrm{loc}}(\mathbb{R}^n, \{0,1\}) \mapsto P(E, \Omega)$ is $L^1_{\mathrm{loc}}(\mathbb{R}^n)$-lower semicontinuous for any $\Omega \in \mathrm{Op}(\mathbb{R}^n)$, the map $\mathcal{A} \in \mathbb{P}(N) \mapsto \mathrm{Per}_{\Phi}(\mathcal{A}, \Omega)$ is $L^1_{\mathrm{loc}}(\mathbb{R}^n)$-lower semicontinuous. The following compactness result can be proven using [11, Theorem 3.39] and a diagonal argument.

Theorem 4.2 (**Compactness).** *Let* $\{\mathcal{A}^{(l)}\} \subset \mathbb{P}(N)$ *be a sequence of partitions such that*

$$\sup_{l \ge 1} \mathrm{Per}\,(\mathcal{A}^{(l)}, \Omega) < +\infty \qquad \forall \Omega \in \mathrm{Op}_b(\mathbb{R}^n).$$

Then there exist a partition $\mathcal{A} \in \mathbb{P}(N)$ *and a subsequence* $\{\mathcal{A}^{(l_k)}\}$ *such that* $\mathcal{A}^{(l_k)}$ *converges to* $\mathcal{A}$ *in* $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ *as* $k \to +\infty$.

The next result is proven for the convenience of the reader.

Proposition 4.3 (**Boundaries of "neighboring" sets).** *Let* $\mathcal{A} \in \mathbb{P}(N)$. *Then*

$$\mathcal{H}^{n-1}\Big(\partial^* A_i \setminus \bigcup_{j=1,\, j\neq i}^{N} \partial^* A_j\Big) = 0 \qquad \forall i = 1, \dots, N.$$

Proof. The case $N = 2$ is classical, so we suppose $N \geq 3$. It is enough to consider $i = 1$. Set

$$\Sigma^{(r)} := \partial^* A_1 \cap \Big(\bigcup_{j=2}^{N} \partial^* A_j \Big), \quad \Sigma^{(s)} := \partial^* A_1 \setminus \Sigma^{(r)}.$$

We divide the proof into four steps.

Step 1. If $x \in \partial^* A_1$ then there exists at most one $j \in \{2, \ldots, N\}$ such that $x \in \partial^* A_j$. Indeed, otherwise up to a relabelling we would have $x \in \partial^* A_1 \cap \partial^* A_2 \cap \partial^* A_3$ and hence $\sum_{j=1}^{3} \frac{|A_j \cap B_r|}{|B_r|} \leq 1$, where $B_r := B_r(x)$. Now by Theorem 1.2 we get $1 \geq \sum_{j=1}^{3} \lim_{r \to 0^+} \frac{|A_j \cap B_r|}{|B_r|} = \frac{3}{2}$, a contradiction.

Step 2. If there exists a unique $j \in \{2, \ldots, N\}$ so that $x \in \partial^* A_1 \cap \partial A_j$, then $x \in \partial^* A_j$.

Indeed, since ∂A_k is closed, and $x \notin \partial A_k$ there is $\rho > 0$ such that $\mathrm{dist}(x, \partial A_k) \geq \rho$ for every $k \neq 1, j$, Hence, for every $r \in (0, \rho)$ up to an $\mathcal{L}^n$-negligible set we have $B_r = (A_1 \cup A_j) \cap B_r$ (and $A_1 \cap A_j = \emptyset$). Thus, $P(A_1, B_r) = P(A_j, B_r)$ for all $r \in (0, \rho)$, and since $x \in \partial^* A_1$,

$$\nu_{A_j}(x) := -\lim_{r \to 0^+} \frac{D\chi_{A_j}(B_r)}{P(A_j, B_r)} = \lim_{r \to 0^+} \frac{D\chi_{A_1}(B_r)}{P(A_1, B_r)} = -\nu_{A_1}(x),$$

hence $|\nu_{A_j}(x)| = 1$. This yields $x \in \partial^* A_j$.

Step 3. If $x \in \Sigma^{(s)}$, there are at least two indices $2 \leq k < l \leq N$ such that $x \in \partial A_k \cap \partial A_l$. Indeed, since $x \in \partial A_1$, there exists at least one $k \in \{2, \ldots, N\}$ such that $x \in \partial A_k$. If k is unique with this property, by Step 2 $x \in \partial^* A_k$ and hence, by definition $x \in \Sigma^{(r)}$.

Step 4. Now we prove $\mathcal{H}^{n-1}(\Sigma^{(s)}) = 0$. We may suppose that $\Sigma^{(s)}$ is bounded, otherwise we consider $\Sigma^{(s)} \cap B_R(0)$ and then let $R \to +\infty$.

By Steps 2 and 3, $x \in \Sigma^{(s)}$ if and only if $x \in (\partial A_i \setminus \partial^* A_i) \cap (\partial A_j \setminus \partial^* A_j)$ for some $i > j > 1$, therefore $\Sigma^{(s)} \subseteq \bigcup_{j=2}^{N} (\partial A_j \setminus \partial^* A_j)$ and $\sum_{j=2}^{N} P(A_j, \Sigma^{(s)}) = 0$. Hence for every $\varepsilon > 0$ there exists an open set $U \subseteq \mathbb{R}^n$ such that $\Sigma^{(s)} \subseteq U$ and $\sum_{j=2}^{N} P(A_j, U) < \varepsilon$. Since $\Sigma^{(s)} \subseteq \partial^* A_1$, by Theorem 1.2 for every $x \in \Sigma^{(s)}$, $r^{1-n} P(A_1, B_r(x)) \to \omega_{n-1}$ as $r \to 0^+$, thus there exists $\rho(x) > 0$ such that

$$\frac{\omega_{n-1}}{2} \leq \frac{P(A_1, B_r(x))}{r^{n-1}} \leq 2\omega_{n-1} \qquad \forall r \in (0, \rho(x)). \tag{4.2}$$

Fix $\delta > 0$ and consider the collection of balls $\mathtt{F} := \{B_r(x) : x \in \Sigma^{(s)}, r \in (0, \min\{\delta, \rho(x)\}), B_r(x) \subset U\}$. Clearly, this is a fine cover of $\Sigma^{(s)}$ and hence by Vitali Covering Lemma there exists an at most countable disjoint subfamily $\mathtt{F}' \subseteq \mathtt{F}$ with $\Sigma^{(s)} \subseteq \bigcup_{B_{r_k} \in \mathtt{F}'} B_{5r_k}$.

Now using (4.2), the definition of partition and (1.2) for the Hausdorff premeasures we get

$$\mathcal{H}^{n-1}_{10\delta}(\Sigma^{(s)}) \le \sum_{B_{r_k}\in \mathrm{F}'} \omega_{n-1}(5r_k)^{n-1} = 2\cdot 5^{n-1}\sum_{B_{r_k}\in \mathrm{F}'} \frac{\omega_{n-1}}{2}\, r_k^{n-1} \le 2\cdot 5^{n-1}\sum_{B_{r_k}\in \mathrm{F}'} P(A_1, B_{r_k})$$

$$=2\cdot 5^{n-1}P\Big(A_1, \bigcup_{B_{r_k}\in \mathrm{F}'} B_{r_k}\Big) \le 2\cdot 5^{n-1}P(A_1,U) = 2\cdot 5^{n-1}P\Big(\bigcup_{j=2}^N A_j, U\Big)$$

$$\le 2\cdot 5^{n-1}\sum_{j=2}^N P(A_j,U) < 2\cdot 5^{n-1}\varepsilon.$$

Thus, letting $\delta,\varepsilon\to 0^+$, we establish $\mathcal{H}^{n-1}(\Sigma^{(s)})=0$. □

Remark 4.4. From Proposition 4.3 it follows that

$$\mathrm{Per}\,(\mathcal{A},\Omega) = \sum_{j=1}^N \mathcal{H}^{n-1}(\Omega\cap\partial^* A_j) = 2\sum_{1\le i<j\le N}\mathcal{H}^{n-1}(\Omega\cap\partial^*A_i\cap\partial^*A_j).$$

Since $\mathcal{H}^{n-1}(\Omega\cap\partial^*A_j\cap\partial^*A_i)$ is the area of the interface between the phases A_i and A_j, $\frac12\,\mathrm{Per}\,(\mathcal{A},\Omega)$ measures the total perimeter of the interfaces in Ω.

4.1.1 Almost minimizers for anisotropic perimeter of partitions

Definition 4.5 ((Λ, r_0) **-minimizers).** *Given* $\Phi=(\phi_1,\dots,\phi_N)$, $\Lambda_1,\Lambda_2\ge 0$, $\alpha_1,\alpha_2>\frac{n-1}{n}$ *and* $r_0\in(0,+\infty]$ *we say that a partition* $\mathcal{A}\in\mathbb{P}(N)$ *is a* $(\Phi,\Lambda_1,\Lambda_2,r_0,\alpha_1,\alpha_2)$*-minimizer of* Per_Φ *in* $\mathbb{R}^n$ *(a* $(\Phi,\Lambda_1,\Lambda_2,r_0,\alpha_1,\alpha_2)$*-minimizer, for short) if*

$$\mathrm{Per}_\Phi(\mathcal{A},B_r)\le \mathrm{Per}_\Phi(\mathcal{B},B_r)+\Lambda_1|\mathcal{A}\Delta\mathcal{B}|^{\alpha_1}+\Lambda_2|\mathcal{A}\Delta\mathcal{B}|^{\alpha_2}$$

whenever $\mathcal{B}\in\mathbb{P}(N)$, $\mathcal{A}\Delta\mathcal{B}\subset\subset B_r$, *and* $r\in(0,r_0)$.

Theorem 4.6 (Density estimates for almost minimizers). *Let* $\mathcal{A}\in\mathbb{P}(N)$ *be a* $(\Phi,\Lambda_1,\Lambda_2,r_0,\alpha_1,\alpha_2)$*-minimizer in* $\mathbb{R}^n$ *and let* $i\in\{1,\dots,N\}$. *Then either* $A_i=\emptyset$ *or for any* $x\in\partial A_i$, $r\in(0,\hat r_0)$, *where*[1] $\hat r_0 := \min\Big\{r_0, \Big(\frac{\kappa_1 n}{2^{1+\alpha_1}\omega_n^{\alpha_1-1}\Lambda_1}\Big)^{\frac{1}{n\alpha_1-n+1}}, \Big(\frac{\kappa_1 n}{2^{1+\alpha_2}\omega_n^{\alpha_2-1}\Lambda_2}\Big)^{\frac{1}{n\alpha_2-n+1}}\Big\}$,

$$\frac{|A_i\cap B_r(x)|}{|B_r(x)|}\le 1-\Big(\frac{\kappa_1}{2(\kappa_1+\kappa_2)}\Big)^n,\qquad \frac{P(A_i,B_r(x))}{r^n}\le\Big(\frac{\kappa_2}{\kappa_1}+\frac12\Big)\,n\omega_n. \tag{4.3}$$

Moreover, if $\kappa:=(N+1)\kappa_1-(N-1)\kappa_2>0$, *then there exists* $c(n,N,\kappa_1,\kappa_2)\in(0,1)$ *such that for every* $r\in(0,\tilde r_0)$, *where* $\tilde r_0:=\min\Big\{r_0,\Big(\frac{\kappa n}{2^{2+\alpha_1}(N-1)\Lambda_1\omega_n^{\alpha_1-1}}\Big)^{\frac{1}{n\alpha_1-n+1}},\Big(\frac{\kappa n}{2^{2+\alpha_2}(N-1)\Lambda_2\omega_n^{\alpha_1-1}}\Big)^{\frac{1}{n\alpha_2-n+1}}\Big\}$, *the following density estimates hold:*

$$\Big(\frac{\kappa}{2(N+1)\kappa_1+2(N-1)\kappa_2}\Big)^n\le\frac{|A_i\cap B_r(x)|}{|B_r(x)|},\qquad c(n,N,\kappa_1,\kappa_2)\le\frac{P(A_i,B_r(x))}{r^n}. \tag{4.4}$$

[1]We suppose $1/\Lambda=+\infty$ if $\Lambda=0$.

Remark 4.7. $\kappa > 0$ occurs, for example, when $\kappa_1 = \kappa_2$.

Proof of Theorem 4.6. We may suppose that $i = 1$ and $A_i \neq \emptyset$. Since $\overline{\partial^* A_1} = \partial A_1$, (4.3)-(4.4) are enough to be shown for $x \in \partial^* A_1$. Set $B_r := B_r(x)$. Choose $r \in (0, \hat{r}_0)$ such that

$$\sum_{j=1}^{N} \mathcal{H}^n(\partial B_r \cap \partial^* A_j) = 0$$

and define the partition $\mathcal{B} := (A_1 \cap B_r, A_2 \setminus B_r, \ldots, A_N \setminus B_r)$. Then $\mathcal{A}\Delta\mathcal{B} \subset\subset B_s$ for every $s \in (r, \hat{r}_0)$ and thus, by minimality and the essential disjointness of A_j,

$$\begin{aligned} 0 \leq &\operatorname{Per}_\Phi(\mathcal{B}, B_s) - \operatorname{Per}_\Phi(\mathcal{A}, B_s) + \Lambda_1|\mathcal{A}\Delta\mathcal{B}|^{\alpha_1} + \Lambda_2|\mathcal{A}\Delta\mathcal{B}|^{\alpha_2} \\ = &P_{\phi_1}(A_1 \cup B_r, B_s) - P_{\phi_1}(A_1, B_s) + \sum_{j=2}^{N} \Big(P_{\phi_j}(A_j \setminus B_r, B_s) - P_{\phi_j}(A_j, B_s) \Big) \\ &+ 2^{\alpha_1}\Lambda_1|B_r \setminus A_1|^{\alpha_1} + 2^{\alpha_2}\Lambda_2|B_r \setminus A_1|^{\alpha_2}. \end{aligned} \tag{4.5}$$

Since

$$\begin{aligned} P_{\phi_1}(A_1 \cup B_r, B_s) =&P_{\phi_1}(A_1, B_s \setminus \overline{B_r}) + \int_{A_1^{(0)} \cap \partial B_r} \phi_1(\nu_{B_r}) d\mathcal{H}^{n-1}, \\ P_{\phi_j}(A_j \setminus B_r, B_s) =&P_{\phi_j}(A_j, B_s \setminus \overline{B_r}) + \int_{A_j^{(1)} \cap \partial B_r} \phi_j(\nu_{B_r}) d\mathcal{H}^{n-1}, \end{aligned} \tag{4.6}$$

from (4.1) we deduce

$$\begin{aligned} \sum_{j=2}^{N} P_{\phi_j}(A_j \setminus B_r, B_s) &= \sum_{j=2}^{N} P_{\phi_j}(A_j, B_s \setminus \overline{B_r}) + \sum_{j=2}^{N} \int_{A_j^{(1)} \cap \partial B_r} \phi_j(\nu_{B_r}) d\mathcal{H}^{n-1} \\ &\leq \sum_{j=2}^{N} P_{\phi_j}(A_j, B_s \setminus \overline{B_r}) + \kappa_2 \sum_{j=2}^{N} \mathcal{H}^{n-1}(A_j^{(1)} \cap \partial B_r). \end{aligned}$$

By Corollary 1.3 and by choice of r

$$\sum_{j=2}^{N} \mathcal{H}^{n-1}(A_j^{(1)} \cap \partial B_r) = \mathcal{H}^{n-1}(A_1^{(0)} \cap \partial B_r) = \mathcal{H}^{n-1}(A_1^c \cap \partial B_r),$$

and hence, from (4.5) we get

$$\sum_{j=1}^{N} P_{\phi_j}(A_j, B_r) \leq 2\kappa_2 \mathcal{H}^{n-1}(A_1^c \cap \partial B_r) + 2^{\alpha_1}\Lambda_1|A_1^c \cap B_r|^{\alpha_1} + 2^{\alpha_2}\Lambda_2|A_1^c \cap B_r|^{\alpha_2}.$$

On the other hand, from (4.1), (1.2) and the essential disjointness of A_j we get

$$\sum_{j=2}^{N} P_{\phi_j}(A_j, B_r) \geq \kappa_1 \sum_{j=2}^{N} P(A_j, B_r) \geq \kappa_1 P\Big(\bigcup_{j=2}^{N} A_j, B_r\Big) = \kappa_1 P(A_1^c, B_r) = \kappa_1 P(A_1, B_r)$$

and thus $\sum_{j=1}^{N} P_{\phi_j}(A_j, B_r) \ge 2\kappa_1 P(A_1, B_r)$. Therefore,

$$\kappa_1 P(A_1^c, B_r) \le \kappa_2 \mathcal{H}^{n-1}(A_1^c \cap \partial B_r) + 2^{\alpha_1 - 1}\Lambda_1 |A_1^c \cap B_r|^{\alpha_1} + 2^{\alpha_2-1}\Lambda_2 |A_1^c \cap B_r|^{\alpha_2}. \tag{4.7}$$

Adding $\kappa_1 \mathcal{H}^{n-1}(A_1^c \cap \partial B_r)$ to both sides of (4.7) and using $\mathcal{H}^{n-1}(\partial B_r \cap \partial^* A_1) = 0$ we establish

$$\kappa_1 P(A_1^c \cap B_r) \le (\kappa_1 + \kappa_2)\mathcal{H}^{n-1}(A_1^c \cap \partial B_r) + 2^{\alpha_1 - 1}\Lambda_1 |A_1^c \cap B_r|^{\alpha_1} + 2^{\alpha_2-1}\Lambda_2 |A_1^c \cap B_r|^{\alpha_2}.$$

Now by the isoperimetric inequality [45]

$$\kappa_1 n\omega_n^{1/n} |A_1^c \cap B_r|^{\frac{n-1}{n}} \le (\kappa_1 + \kappa_2)\mathcal{H}^{n-1}(A_1^c \cap \partial B_r) + 2^{\alpha_1 - 1}\Lambda_1 |A_1^c \cap B_r|^{\alpha_1} + 2^{\alpha_2-1}\Lambda_2 |A_1^c \cap B_r|^{\alpha_2}. \tag{4.8}$$

By choice of $\hat r_0$ we have

$$2^{\alpha_k - 1}\Lambda_k |A_1^c \cap B_r|^{\alpha_k - \frac{n-1}{n}} \le 2^{\alpha_k-1}\Lambda_k \omega_n^{\alpha_k - \frac{n-1}{n}} \hat r_0^{n\alpha_k - n + 1} \le \frac{\kappa_1 n \omega_n^{1/n}}{4}, \quad k = 1, 2.$$

As a result from (4.8) we obtain

$$\frac{\kappa_1}{2(\kappa_1+\kappa_2)} n\omega_n^{1/n} |A_1^c \cap B_r|^{\frac{n-1}{n}} \le \mathcal{H}^{n-1}(A_1^c \cap \partial B_r),$$

and whence

$$|A_1^c \cap B_r| \ge \left(\frac{\kappa_1}{2(\kappa_1+\kappa_2)}\right)^n \omega_n r^n,$$

i.e.

$$\frac{|A_1 \cap B_r|}{|B_r|} \le 1 - \left(\frac{\kappa_1}{2(\kappa_1+\kappa_2)}\right)^n.$$

From (4.7) and by the definition of $\hat r_0$ we obtain the upper perimeter density estimates for a.e. r :

$$P(A_1, B_r) \le \frac{\kappa_2}{\kappa_1}\mathcal{H}^{n-1}(\partial B_r) + \frac{2^{\alpha_1-1}\Lambda_1}{\kappa_1}|B_r|^{\alpha_1} + \frac{2^{\alpha_2-1}\Lambda_2}{\kappa_1}|B_r|^{\alpha_2} \le \left(\frac{\kappa_2}{\kappa_1} + \frac{1}{2}\right) n\omega_n r^{n-1}.$$

As $r \to P(A_1, B_r)$ is continuous from the left, this extends for all $r \in (0, \hat r_0)$.

Now supposing $\kappa > 0$, let us prove the lower volume density estimate. As above we assume $i = 1$ and $A_1 \neq \emptyset$. Take $x \in \partial^* A_1$ and let $r \in (0, \tilde r_0)$ be such that

$$\sum_{j=1}^{N} \mathcal{H}^{n-1}(\partial^* A_j \cap \partial B_r) = 0.$$

Set

$$I_1 := \{j \in \{2, \dots, N\} : \ \mathcal{H}^{n-1}(B_{\tilde r_0}(x) \cap \partial^* A_1 \cap \partial^* A_j| > 0\}.$$

Since $x \in \partial A_1$, one has $I_1 \neq \emptyset$. For every $j \in I_1$ let us define

$$\mathcal{B}^{(j)} := (A_1 \setminus B_r, A_2, \dots, A_j \cup (A_1 \cap B_r), \dots, A_N).$$

By the minimality of $\mathcal{A}$ for every $s \in (r, \tilde{r}_0)$ one has

$$
\begin{aligned}
P_{\phi_1}(A_1, B_s) + P_{\phi_j}(A_j, B_s) \le & P_{\phi_1}(A_1 \setminus B_r, B_s) + P_{\phi_j}(A_j \cup (A_1 \cap B_r), B_s) \\
& + 2^{\alpha_1}\Lambda_1 |A_1 \cap B_r|^{\alpha_1} + 2^{\alpha_2}\Lambda_2 |A_1 \cap B_r|^{\alpha_2}.
\end{aligned}
$$

Using

$$
\begin{aligned}
P_{\phi_j}(A_j \cup (A_1 \cap B_r), B_s) = & P_{\phi_j}(A_j, B_s) + P_{\phi_j}(A_1, B_r) + \int_{A_1 \cap \partial B_r} \phi_j(\nu_{B_r}) d\mathcal{H}^{n-1} \\
& - \int_{B_r \cap \partial^* A_1 \cap \partial^* A_j} \big(\phi_j(\nu_{A_1}) + \phi_j(\nu_{A_j})\big) d\mathcal{H}^{n-1}
\end{aligned}
$$

and (4.6) we obtain

$$
\begin{aligned}
\int_{B_r \cap \partial^* A_1 \cap \partial^* A_j} \big(\phi_j(\nu_{A_1}) + \phi_j(\nu_{A_j})\big) d\mathcal{H}^{n-1} \le & P_{\phi_j}(A_1, B_r) - P_{\phi_1}(A_1, B_r) \\
& + \int_{A_1 \cap \partial B_r} \Big(\phi_1(\nu_{B_r}) + \phi_j(\nu_{B_r})\Big) d\mathcal{H}^{n-1} + 2^{\alpha_1}\Lambda_1 |A_1 \cap B_r|^{\alpha_1} + 2^{\alpha_2}\Lambda_2 |A_1 \cap B_r|^{\alpha_2}.
\end{aligned}
$$

Summing these inequalities in $j \in I_1$ and using (4.1) we establish

$$
\begin{aligned}
2\kappa_1 \sum_{j=2}^{N} \mathcal{H}^{n-1}(B_r \cap \partial^* A_1 \cap \partial^* A_j) \le & 2\kappa_2 (N-1) \mathcal{H}^{n-1}(A_1 \cap \partial B_r) \\
& + (N-1)(\kappa_2 - \kappa_1) P(A_1, B_r) + (N-1)(2^{\alpha_1}\Lambda_1 |A_1 \cap B_r|^{\alpha_1} + 2^{\alpha_2}\Lambda_2 |A_1 \cap B_r|^{\alpha_2}).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\kappa P(A_1 \cap B_r) \le & ((N+1)\kappa_1 + (N-1)\kappa_2) \mathcal{H}^{n-1}(A_1 \cap \partial B_r) \\
& + (N-1)(2^{\alpha_1}\Lambda_1 |A_1 \cap B_r|^{\alpha_1} + 2^{\alpha_2}\Lambda_2 |A_1 \cap B_r|^{\alpha_2}).
\end{aligned}
$$

By the definition of $\tilde{r}_0$ we have $(N-1)\Lambda_k |A_1 \cap B_r|^{\alpha_k - \frac{n-1}{n}} \le \frac{\kappa n \omega_n^{1/n}}{4}$, $k = 1, 2,$ and thus, by the isoperimetric inequality,

$$
\frac{\kappa n \omega_n^{1/n}}{2(N+1)\kappa_1 + 2(N-1)\kappa_2} |A_1 \cap B_r|^{\frac{n-1}{n}} \le \mathcal{H}^{n-1}(A_1 \cap \partial B_r).
$$

Now integrating we get

$$
\left(\frac{\kappa}{2(N+1)\kappa_1 + 2(N-1)\kappa_2} \right)^n \omega_n r^n \le |A_1 \cap B_r|.
$$

The lower perimeter density estimates follows from the volume density estimates and the relative isoperimetric inequality for the ball. $\square$

4.1.2 Bounded partitions

The multiphase analog of a bounded phase in $\mathbb{R}^n$ is the following.

Definition 4.8 (Bounded partition). *A partition $\mathcal{C} = (C_1, \dots, C_{N+1}) \in \mathbb{P}(N+1)$ is called* **bounded** *if C_i is bounded for each $i = 1, \dots, N$.*

Therefore, C_{N+1} is the only unbounded entry of $\mathcal{C}$. We denote by $\mathbb{P}_b(N+1)$ the collection of all bounded partitions of $\mathbb{R}^n$.

Given $\mathcal{A} \in \mathbb{P}_b(N+1)$, we denote by

$$\mathrm{co}(\mathcal{A})$$

the closed convex hull of $\bigcup_{i=1}^{N} A_i$. Since $\mathcal{A}\Delta\mathcal{B} \subset\subset \mathbb{R}^n$ for every $\mathcal{A}, \mathcal{B} \in \mathbb{P}_b(N+1)$,

$$|\mathcal{A}\Delta\mathcal{B}| = \sum_{j=1}^{N+1} |A_j \Delta B_j|$$

is the $L^1(\mathbb{R}^n)$-distance in $\mathbb{P}_b(N+1)$.

The following compactness result can be proven similarly to Theorem 4.2.

Theorem 4.9 (Compactness). *Let $\mathcal{A}^{(k)} \in \mathbb{P}_b(N+1)$, $k = 1, 2, \dots,$ and $\Omega \in \mathrm{Op}_b(\mathbb{R}^n)$ be such that*

$$\sup_{k\geq 1} \mathrm{Per}\,(\mathcal{A}^{(k)}) < +\infty, \qquad \mathrm{co}(\mathcal{A}^{(k)}) \subseteq \Omega \qquad \forall k \geq 1.$$

Then there exist $\mathcal{A} \in \mathbb{P}_b(N+1)$ and a subsequence $\{\mathcal{A}^{(k_l)}\}$ converging to $\mathcal{A}$ in $L^1(\mathbb{R}^n)$ as $l \to +\infty$. Moreover, $\bigcup_{i=1}^{N} A_j \subseteq \overline{\Omega}$.

4.2 Existence of generalized minimizing movements for bounded partitions

Given $E, F \subseteq \mathbb{R}^n$ set

$$\bar{\sigma}(E, F) := \int_{E\Delta F} d(x, \partial F)dx.$$

Note that $\bar{\sigma}(E, F) = 0$ if $|E\Delta F| = 0$ whereas $\bar{\sigma}(E, F) = +\infty$ if $\partial F = \emptyset$ and $|E\Delta F| > 0$. Moreover, $X, Y \subseteq \mathbb{R}^n$ are measurable and $\partial Y \neq \emptyset$,

$$\begin{aligned} \int_{X\Delta Y} d(x, \partial Y)dx &= \int_X \tilde{d}(x, \partial Y)dx - \int_Y \tilde{d}(x, \partial Y)dx \quad \text{if } X \cap Y \text{ is bounded,} \\ \int_{X\Delta Y} d(x, \partial Y)dx &= \int_{Y^c} \tilde{d}(x, \partial Y)dx - \int_{X^c} \tilde{d}(x, \partial Y)dx \quad \text{if } X^c \cap Y^c \text{ is bounded.} \end{aligned} \tag{4.9}$$

Now the *nonsymmetric distance* between $\mathcal{A}, \mathcal{B} \in \mathbb{P}_b(N+1)$ is defined as

$$\sigma(\mathcal{A}, \mathcal{B}) := \sum_{i=1}^{N+1} \bar{\sigma}(A_i, B_i),$$

where $N+1 \geq 2$. Observe that for every $\mathcal{B} \in \mathbb{P}_b(N+1)$ the map $\sigma(\cdot, \mathcal{B})$ is $L^1(\mathbb{R}^n)$-lower semicontinuous.

Given an $(N+1)$-tuple $\Phi := (\phi_1, \ldots, \phi_{N+1})$ of anisotropies, $\mathcal{A} \in \mathbb{P}_b(N+1)$ and measurable functions $H_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, N+1$, consider the functional

$$F_H^\Phi(\mathcal{B}, \mathcal{A}; \lambda) = \mathrm{Per}_\Phi(\mathcal{B}) + \lambda \sigma(\mathcal{B}, \mathcal{A}) + \sum_{i=1}^{N+1} \int_{B_i} H_i dx, \qquad \mathcal{B} \in \mathbb{P}_b(N+1).$$

In this section we always suppose

$$\text{the entries of } \Phi \text{ satisfy (4.1) and } \kappa := (N+1)\kappa_1 - (N-1)\kappa_2 > 0,$$

and $H := (H_1, \ldots, H_{N+1})$ satisfies

$$\begin{cases} H_i \in L^p_{\mathrm{loc}}(\mathbb{R}^n),\, i = 1, \ldots, N+1, \text{ for some } p > n \text{ and } H_{N+1} \in L^1(\mathbb{R}^n); \\ \text{there exists } R > 0 \text{ such that } H_i \geq H_{N+1} \text{ a.e. in } \mathbb{R}^n \setminus B_R(0) \text{ for any } i = 1, \ldots, N; \end{cases}$$

in particular $F_H^\Phi(\cdot, \mathcal{A}; \lambda)$ is well-defined and $L^1(\mathbb{R}^n)$-lower semicontinuous.

Notice that the forcing term can also be reresented as

$$\sum_{j=1}^{N+1} \int_{B_j} H_j dx = \sum_{j=1}^{N} \int_{B_j} (H_j - H_{N+1}) dx + \int_{\mathbb{R}^n} H_{N+1} dx.$$

When $\phi_1 = \phi_2$ are Euclidean, $N = 1$ and $H_2 = 0$, we get the Almgren-Taylor-Wang functional with an external force H_1 which is nonnegative outside a sufficiently large ball.

F_H^Φ is the natural generalization of the Almgren-Taylor-Wang functional [6] to the case of partitions [33, 48]. One can readily check that the map $\mathcal{B} \in \mathbb{P}_b(N+1) \mapsto F_H^\Phi(\mathcal{B}, \mathcal{A}; \lambda)$ is $L^1(\mathbb{R}^n)$-lower semicontinuous.

Theorem 4.10 (Existence of minimizers of F_H^Φ). *Given $\mathcal{A} \in \mathbb{P}_b(N+1)$ and $\lambda \geq 1$ the problem*

$$\inf_{\mathcal{B} \in \mathbb{P}_b(N+1)} F_H^\Phi(\mathcal{B}, \mathcal{A}; \lambda) \tag{4.10}$$

has a solution. Moreover, every minimizer $\mathcal{A}(\lambda) = (A_1(\lambda), \ldots, A_{N+1}(\lambda))$ satisfies the bound

$$\bigcup_{i=1}^{N} A_i(\lambda) \subseteq \textit{ closed convex hull of } \mathrm{co}(\mathcal{A}) \cup B_R(0).$$

Proof. Let D stand for the closed convex hull of $\mathrm{co}(\mathcal{A})\cup B_R(0)$. Given a partition $\mathcal{B}\in\mathbb{P}_b(N+1)$ define the competitor $\mathcal{B}'\in\mathbb{P}_b(N+1)$ as

$$\mathcal{B}' := \Big(B_1\cap D,\dots,B_N\cap D,\mathbb{R}^n\setminus\bigcup_{i=1}^N(B_i\cap D)\Big). \tag{4.11}$$

By the comparison theorem 2.9 we have $P_{\phi_i}(B_i)\geq P_{\phi_i}(B_i\cap D)$ for $i=1,\dots,N,$ and

$$\begin{aligned}P_{\phi_{N+1}}(B_{N+1}) =&P_{\phi_{N+1}}\Big(\bigcup_{i=1}^N B_i\Big)\geq P_{\phi_{N+1}}\Big(\Big(\bigcup_{i=1}^N B_i\Big)\cap D\Big)\\ =&P_{\phi_{N+1}}\Big(\bigcup_{i=1}^N(B_i\cap D)\Big) = P_{\phi_{N+1}}\Big(\mathbb{R}^n\setminus\bigcup_{i=1}^N(B_i\cap D)\Big).\end{aligned}$$

In addition, for $i=1,\dots,N$

$$\begin{aligned}\int_{B_i\Delta A_i} d(x,\partial A_i)dx =&\int_{B_i\setminus A_i} d(x,\partial A_i)dx+\int_{A_i\setminus B_i} d(x,\partial A_i)dx\\ \geq&\int_{(B_i\cap D)\setminus A_i} d(x,\partial A_i)dx+\int_{A_i\setminus(B_i\cap D)} d(x,\partial A_i)dx\\ =&\int_{(B_i\cap D)\Delta A_i} d(x,\partial A_i)dx,\end{aligned} \tag{4.12}$$

where we used the nonnegativity of the distance function and $A_i\setminus B_i = A_i\setminus(B_i\cap D)$. The equality in (4.12) holds if and only if $\Big|\bigcup\limits_{i=1}^N B_i\setminus D\Big| = 0$. For the same reason, since $A^c_{N+1}=\bigcup\limits_{i=1}^N A_i\subseteq D,$

$$\begin{aligned}\int_{B_{N+1}\Delta A_{N+1}} d(x,\partial A_{N+1})dx =&\int_{B^c_{N+1}\Delta A^c_{N+1}} d(x,\partial A_{N+1})dx\\ \geq&\int_{(B^c_{N+1}\cap D)\Delta A^c_{N+1}} d(x,\partial A_{N+1})dx.\end{aligned}$$

Finally, since $H_i\geq H_{N+1}$ a.e. in $\mathbb{R}^n\setminus D,$ one has also

$$\begin{aligned}\sum_{j=1}^{N+1}\int_{B_j} H_j dx =&\sum_{j=1}^N\int_{B_j}(H_j-H_{N+1})dx+\int_{\mathbb{R}^n} H_{N+1}dx\\ \geq&\sum_{j=1}^N\int_{B_j\cap D}(H_j-H_{N+1})dx+\int_{\mathbb{R}^n} H_{N+1}dx=\sum_{j=1}^{N+1}\int_{B'_j} H_j dx.\end{aligned}$$

Thus, we have

$$F^{\Phi}_H(\mathcal{B},\mathcal{A};\lambda)\geq F^{\Phi}_H(\mathcal{B}',\mathcal{A};\lambda)\qquad \forall\mathcal{B}\in\mathbb{P}_b(N+1)$$

and the inequality is strict whenever $\Big|\bigcup_{i=1}^{N} B_i \setminus D\Big| > 0$.

Let $\{\mathcal{B}^{(k)}\} \subseteq \mathbb{P}_b(N+1)$ be a minimizing sequence, which can be supposed so that $\mathrm{co}(\mathcal{B}^{(k)}) \subseteq D$ and $F_H^{\Phi}(\mathcal{B}^{(k)}, \mathcal{A}; \lambda) \le F_H^{\Phi}(\mathcal{T}, \mathcal{A}; \lambda)$, $\mathcal{T} := (\emptyset, \dots, \emptyset, \mathbb{R}^n)$, so that

$$\kappa_1 \mathrm{Per}\,(\mathcal{B}^{(k)}) \le \frac{\lambda}{2}\,\sigma(\mathcal{T}, \mathcal{A}) + \sum_{j=1}^{N} \int_D |H_i - H_{N+1}|dx + \int_{\mathbb{R}^n} |H_{N+1}|dx \qquad \forall k \ge 1.$$

By Proposition 4.9 there exists $\mathcal{A}(\lambda) \in \mathbb{P}_b(N+1)$ such that $\mathcal{B}^{(k)} \to \mathcal{A}(\lambda)$ in $L^1(\mathbb{R}^n)$ as $k \to +\infty$. Then the $L^1(\mathbb{R}^n)$-lower semicontinuity of $F_H^{\Phi}(\cdot, \mathcal{A}; \lambda)$ implies that $\mathcal{A}(\lambda)$ is a solution to (4.10).

Now let $\mathcal{A}(\lambda)$ be a minimizer of $F_H^{\Phi}(\cdot, \mathcal{A}; \lambda)$. If $\Big|\bigcup_{j=1}^{N} A_j(\lambda) \setminus D\Big| > 0$, then, as shown above, $F_H^{\Phi}(\mathcal{A}(\lambda), \mathcal{A}; \lambda) > F_H^{\Phi}(\mathcal{A}(\lambda)', \mathcal{A}; \lambda)$, where $\mathcal{A}(\lambda)'$ is defined as in (4.11), which contradicts the minimality of $\mathcal{A}(\lambda)$. □

Remark 4.11. Suppose that $\mathcal{G} \in \mathbb{P}_b(N+1)$ and $G_i = \emptyset$ for some $i \in \{1, \dots, N\}$. Then by definition of $\bar\sigma$ every minimizer $\mathcal{A}(\lambda) \in \mathbb{P}_b(N+1)$ of $F_H^{\Phi}(\cdot, \mathcal{G}; \lambda)$ satisfies $A_i(\lambda) = \emptyset$. In particular, when the entries Φ are Euclidean and $H = (0, \dots, 0)$, for $\mathcal{G} = (G, \emptyset, \dots, \emptyset, \mathbb{R}^n \setminus G)$ the GMM problem for $F_H^{\Phi}(\cdot, \mathcal{G}; \lambda)$ agrees with the GMM problem of the Almgren-Taylor-Wang functional

$$E \in BV(\mathbb{R}^n) \mapsto P(E) + \lambda \int_{E\Delta G} d(x, \partial G)dx. \tag{4.13}$$

Proposition 4.12 (Behaviour of $\mathcal{A}(\lambda)$ as time goes to 0). *Let $\mathcal{A} \in \mathbb{P}_b(N+1)$ be such that $\sum_{j=1}^{N+1} |\overline{A_j} \setminus A_j| = 0$, and $\mathcal{A}(\lambda)$ be a minimizer of $F_H^{\Phi}(\cdot, \mathcal{A}; \lambda)$. Then:*

a) $\lim_{\lambda \to +\infty} |\mathcal{A}(\lambda) \Delta \mathcal{A}| = 0,$

b) $\lim_{\lambda \to +\infty} \mathrm{Per}_{\,\Phi}(\mathcal{A}(\lambda)) = \mathrm{Per}_{\,\Phi}(\mathcal{A}),$

c) $\lim_{\lambda \to +\infty} \lambda\sigma(\mathcal{A}(\lambda), \mathcal{A}) = 0.$

Proof. a) Let D stand for the closed convex hull of $\mathrm{co}(\mathcal{A}) \cup B_R(0)$. Choose any sequence $\lambda_k \to +\infty$. Since $F_H^{\Phi}(\mathcal{A}(\lambda_k), \mathcal{A}; \lambda_k) \le F_H^{\Phi}(\mathcal{A}, \mathcal{A}; \lambda_k) = \mathrm{Per}_{\,\Phi}(\mathcal{A})$, we have $\mathrm{Per}_{\,\Phi}(\mathcal{A}(\lambda)) \le \mathrm{Per}_{\,\Phi}(\mathcal{A})$ and

$$\lim_{k \to +\infty} \sigma(\mathcal{A}(\lambda_k), \mathcal{A}) = 0. \tag{4.14}$$

Moreover, by Theorem 4.10 $\mathrm{co}(\mathcal{A}(\lambda)) \subseteq D$, therefore Proposition 4.9 yields the existence of a subsequence $\{\lambda_{k_l}\}_l$ and of $\mathcal{B} \in \mathbb{P}_b(N+1)$ such that $\mathcal{A}(\lambda_{k_l}) \to \mathcal{B}$ in $L^1(\mathbb{R}^n)$ as $l \to +\infty$. Now the lower semicontinuity of $\sigma(\cdot, \mathcal{A})$ and (4.14) imply $\sigma(\mathcal{B}, \mathcal{A}) = 0$. Then from the assumption on $\mathcal{A}$ we get $\mathcal{A} = \mathcal{B}$. Since λ_k is arbitrary, a) follows.

b) Since $\mathrm{Per}_{\,\Phi}(\mathcal{A}(\lambda)) \le \mathrm{Per}_{\,\Phi}(\mathcal{A})$, from a) we obtain

$$\mathrm{Per}_{\,\Phi}(\mathcal{A}) \le \liminf_{\lambda \to +\infty} \mathrm{Per}_{\,\Phi}(\mathcal{A}(\lambda)) \le \limsup_{\lambda \to +\infty} \mathrm{Per}_{\,\Phi}(\mathcal{A}(\lambda)) \le \mathrm{Per}_{\,\Phi}(\mathcal{A}).$$

c) From a) and b) we have

$$\limsup_{\lambda\to+\infty} \lambda\sigma(\mathcal{A}(\lambda),\mathcal{A}) \le \limsup_{\lambda\to+\infty} \Big(\mathrm{Per}_{\,\Phi}(\mathcal{A}) - \mathrm{Per}_{\,\Phi}(\mathcal{A}(\lambda)) + \sum_{j=1}^{N} \int_{A_j(\lambda)\Delta A_j} |H_j - H_{N+1}|dx\Big) = 0.$$

□

Theorem 4.13 (Density estimates). *Suppose that $\mathcal{A}\in\mathbb{P}_b(N+1)$ and let $\mathcal{A}(\lambda)\in\mathbb{P}_b(N+1)$ be a minimizer of $F_H^\Phi(\cdot,\mathcal{A};\lambda)$. Then for every $i\in\{1,\dots,N+1\}$ either $\partial A_i(\lambda)$ is empty or there exists $c(N,n,\kappa_1,\kappa_2)\in(0,1)$ such that*

$$\Big(\frac{\kappa}{2(N+2)\kappa_1+2N\kappa_2}\Big)^n \le \frac{|A_i(\lambda)\cap B_r(x)|}{|B_r(x)|} \le 1-\Big(\frac{\kappa_1}{2(\kappa_1+\kappa_2)}\Big)^n, \tag{4.15}$$

$$c(n,N,\kappa_1,\kappa_2) \le \frac{P(A_i(\lambda),B_r(x))}{r^{n-1}} \le \left(\frac{\kappa_2}{\kappa_1}+\frac{1}{2}\right) n\omega_n$$

for any $x\in\partial A_i(\lambda)$ and $r\in\Big(0,\min\Big\{1,\frac{\kappa n}{8N\Lambda_1},\Big(\frac{\kappa n\omega_n^{1/p}}{2^{3-1/p}N\Lambda_2}\Big)^{\frac{p}{p-n}}\Big\}\Big)$, where

$$\Lambda_1 := \lambda(\mathrm{diam}\,D+2), \qquad \Lambda_2 := N^{1/p}\max_{1\le j\le N}\|H_j-H_{N+1}\|_{L^p(D)}.$$

and D is the closed convex hull of $\mathrm{co}(\mathcal{A})\cup B_R(0)$. Moreover

$$\sum_{j=1}^{N+1}\mathcal{H}^{n-1}(\partial A_j(\lambda)\setminus\partial^* A_j(\lambda)) = 0. \tag{4.16}$$

Proof. For every $x\in\mathbb{R}^n$ and $\mathcal{C}\in\mathbb{P}_b(N+1)$ such that $\mathcal{C}\Delta\mathcal{A}(\lambda)\subset\subset B_\rho(x)$ with $\rho\in(0,1)$, the minimality of $\mathcal{A}(\lambda)$ implies

$$\mathrm{Per}_{\,\Phi}(\mathcal{A}(\lambda),B_\rho(x)) \le \mathrm{Per}_{\,\Phi}(\mathcal{C},B_\rho(x)) + \lambda\sum_{j=1}^{N+1}\int_{C_j\Delta A_j(\lambda)} d(x,\partial A_j)dx + \sum_{j=1}^{N}\int_{C_j\Delta A_j(\lambda)}|H_j-H_{N+1}|dx.$$

Since $\mathrm{co}(\mathcal{A}(\lambda))\subseteq D$,

$$d(z,\partial A_j)\le \mathrm{diam}\,D+2\rho \qquad \forall i=j,\dots,N+1,\ z\in\mathcal{C}\Delta\mathcal{A}(\lambda).$$

Therefore as $\mathcal{C}\Delta\mathcal{A}(\lambda)\subseteq B_1$,

$$\sum_{j=1}^{N+1}\int_{C_j\Delta A_j(\lambda)} d(x,\partial A_j)\,dx \le (\mathrm{diam}\,D+2)|\mathcal{C}\Delta\mathcal{A}(\lambda)|$$

and

$$\begin{aligned}\sum_{j=1}^{N}\int_{C_j\Delta A_j(\lambda)}|H_j-H_{N+1}|dx \le& \sum_{j=1}^{N}|C_j\Delta A_j(\lambda)|^{1-1/p}\|H_j-H_{N+1}\|_{L^p(D)}\\ \le& N^{1/p}\max_{1\le j\le N}\|H_j-H_{N+1}\|_{L^p(D)}\,|\mathcal{C}\Delta\mathcal{A}(\lambda)|^{1-1/p}.\end{aligned}$$

Thus,

$$\mathrm{Per}_{\Phi}(\mathcal{A}(\lambda), B_\rho(x)) \le \mathrm{Per}_{\Phi}(\mathcal{C}, B_\rho(x)) + \Lambda_1|\mathcal{C}\Delta\mathcal{A}(\lambda)| + \Lambda_2|\mathcal{C}\Delta\mathcal{A}(\lambda)|^{1-1/p},$$

i.e.

$$\mathcal{A}(\lambda) \text{ is a } (\Phi,\Lambda_1,\Lambda_2,1,1-1/p)\text{-minimizer.}$$

Now application of Theorem 4.6 to $\mathcal{A}(\lambda)$ finishes the proof.

Note that (4.16) follows from the density estimates and the standard covering argument. □

Proposition 4.14. *Suppose that the entries of $\mathcal{A}$ satisfy the density estimates*

$$\theta \le \frac{|A_j\cap B_r(x)|}{|B_r|} \le 1-\theta, \qquad r\in(0,r_0),$$

whenever $x\in\partial A_j$, $j=1,\dots,N+1$. Then for any $\ell > r_0$ and any minimizer $\mathcal{A}(\lambda)$ of $F_H^\Phi(\cdot,\mathcal{A};\lambda)$ one has

$$|\mathcal{A}(\lambda)\Delta\mathcal{A}| \le \frac{3^n\omega_n^{1/n}}{c_{\mathrm{ball}}(n)\theta^{1-1/n}\kappa_1}\left(\frac{\ell}{r_0}\right)^{n-1}\mathrm{Per}_{\Phi}(\mathcal{A})\ell + \frac{1}{\ell}\,\sigma(A_j(\lambda),A_j). \tag{4.17}$$

where $c_{\mathrm{ball}}(n)\in(0,1)$ is the Euclidean relative isoperimetric contant.

Proof. Fix $j\in\{1,\dots,N+1\}$ and define

$$E_j := \{x\in A_j(\lambda)\Delta A_j:\ d(x,\partial A_j)\le \ell\},\quad F_j := \{x\in A_j(\lambda)\Delta A_j:\ d(x,\partial A_j)\ge \ell\}.$$

Clearly

$$|F_j| \le \frac{1}{\ell}\int_{F_j} d(x,\partial E_0)dx \le \frac{1}{\ell}\,\bar\sigma(A_j(\lambda),A_j).$$

Let us estimate $|E_j|$. By a simple covering argument, one can find a finite family of disjoint balls $\{B_\ell(x_k)\}$ $x_k\in\partial A_j$, $k=1,\dots,m$, such that E_j is covered by the family $\{B_{3\ell}(x_k)\}_{k=1}^m$. Then by the density estimates, the relative isoperimetric inequality for balls and the disjointness of $\{B_\rho(x_k)\}$ we get

$$\begin{aligned}
|E_j| &\le \sum_{k=1}^m \omega_n(3\ell)^n = 3^n\omega_n^{1/n} r_0\left(\frac{\ell}{r_0}\right)^n\sum_{k=1}^m(\omega_n r_0^n)^{\frac{n-1}{n}}\\
&\le \frac{3^n\omega_n^{1/n}r_0}{\theta^{1-1/n}}\left(\frac{\ell}{r_0}\right)^n\sum_{k=1}^m\Big(\min\{|B_{r_0}(x_k)\cap A_j|,|B_{r_0}(x_k)\setminus A_j|\}\Big)^{\frac{n-1}{n}}\\
&\le \frac{3^n\omega_n^{1/n}r_0}{c_{\mathrm{ball}}(n)\theta^{1-1/n}}\left(\frac{\ell}{r_0}\right)^n\sum_{k=1}^m P(A_j,B_{r_0}(x_k)) \le \frac{3^n\omega_n^{1/n}r_0}{c_{\mathrm{ball}}(n)\theta^{1-1/n}}\left(\frac{\ell}{r_0}\right)^n P\Big(A_j,\bigcup_{k=1}^m B_{r_0}(x_k)\Big)\\
&\le \frac{3^n\omega_n^{1/n}}{c_{\mathrm{ball}}(n)\theta^{1-1/n}}\left(\frac{\ell}{r_0}\right)^{n-1}P(A_j)\ell \le \frac{3^n\omega_n^{1/n}}{c_{\mathrm{ball}}(n)\theta^{1-1/n}\kappa_1}\left(\frac{\ell}{r_0}\right)^{n-1}P_{\phi_j}(A_j)\ell.
\end{aligned}$$

Thus

$$|A_j(\lambda)\Delta A_j| \le \frac{3^n\omega_n^{1/n}}{c_{\mathrm{ball}}(n)\theta^{1-1/n}\kappa_1}\left(\frac{\ell}{r_0}\right)^{n-1}P_{\phi_j}(A_j)\ell + \frac{1}{\ell}\,\bar\sigma(A_j(\lambda),A_j).$$

Summing over j we get (4.17). □

Remark 4.15. The density estimates show that the entries of $\mathcal{A}(\lambda)$ are Lebesgue-equivalent to open sets. Indeed, since using $\overline{E}\setminus E \subset \partial E$, and $\overline{E}\setminus \overset{\circ}{\overline{E}} \subset \partial E$ ($\overset{\circ}{E}$ being the interior of E), we have

$$\sum_{j=1}^{N+1} |A_j(\lambda)\Delta\overline{\overset{\circ}{A_j}(\lambda)}| \le \sum_{j=1}^{N+1} |\overline{A_j(\lambda)}\setminus A_j(\lambda)| + \sum_{j=1}^{N+1} |\overline{A_j(\lambda)}\setminus \overline{\overset{\circ}{A_j}(\lambda)}| \le 2\sum_{j=1}^{N+1} |\partial A_j(\lambda)|.$$

Now by the density estimates $\sum\limits_{j=1}^{N+1} |\partial A_j(\lambda)| = 0,$ and therefore $\sum\limits_{j=1}^{N+1} |A_j(\lambda)\Delta\overline{\overset{\circ}{A_j}(\lambda)}| = 0.$

One of the main results of the present chapter reads as follows.

Theorem 4.16 (Existence of GMM**).** *Let* $\mathcal{G}\in\mathbb{P}_b(N+1)$. *Then* $GMM(F_H^\Phi,\mathcal{G})$ *is non empty. Moreover, there exists a constant* $\mathrm{C}=\mathrm{C}(N,n,\Phi,H,\mathcal{G})>0$ *such that for any* $\mathcal{M}\in GMM(F,\mathcal{G})$,

$$|\mathcal{M}(t)\Delta\mathcal{M}(t')| \le \mathrm{C}\,|t-t'|^{\frac{1}{n+1}} \qquad \forall t,t'>0,\ |t-t'|<1 \tag{4.18}$$

and

$$\bigcup_{j=1}^{N} M_j(t) \subseteq \textit{closed convex hull of } \mathrm{co}(\mathcal{G})\cup B_R \qquad \forall t\ge 0. \tag{4.19}$$

In addition, if $\sum\limits_{j=1}^{N+1} |\overline{G_j}\setminus G_j| = 0,$ *then* (4.18) *holds for any* $t,t'\ge 0$ *and* $|t-t'|<1.$

Proof. Let D stand for the closed convex hull of $\mathrm{co}(\mathcal{G})\cup B_R$. Given $\lambda\ge 1$ and $k\in\mathbb{N}_0$ we define $\mathcal{G}(\lambda,k)$ recursively as: $\mathcal{G}(\lambda,0)=\mathcal{G}$,

$$F_H^\Phi(\mathcal{G}(\lambda,k+1),\mathcal{G}(\lambda,k);\lambda) = \min_{\mathcal{A}\in\mathbb{P}_b(N+1)} F_H^\Phi(\mathcal{A},\mathcal{G}(\lambda,k);\lambda);$$

recall that existence of minimizers follows from Theorem 4.10. Since $F_H^\Phi(\mathcal{G}(\lambda,k+1),\mathcal{G}(\lambda,k);\lambda) \le F_H^\Phi(\mathcal{G}(\lambda,k),\mathcal{G}(\lambda,k);\lambda),$ we have

$$\begin{aligned}\mathrm{Per}_\Phi(\mathcal{G}(\lambda,k)) &+ \sum_{j=1}^{N}\int_{G_j(\lambda,k)}(H_j-H_{N+1})dx + \lambda\sigma(\mathcal{G}(\lambda,k),\mathcal{G}(\lambda,k-1))\\ &\le \mathrm{Per}_\Phi(\mathcal{G}(\lambda,k-1)) + \sum_{j=1}^{N}\int_{G_j(\lambda,k-1)}(H_j-H_{N+1})dx.\end{aligned} \tag{4.20}$$

Thus, the map

$$k\in\mathbb{N}_0 \mapsto \mathrm{Per}_\Phi(\mathcal{G}(\lambda,k)) + \sum_{j=1}^{N}\int_{G_j(\lambda,k)}(H_j-H_{N+1})dx$$

is non-increasing for any $\lambda\ge 1.$ In particular,

$$\begin{aligned}\mathrm{Per}_\Phi(\mathcal{G}(\lambda,k)) &\le \mathrm{Per}_\Phi(\mathcal{G}(\lambda,0)) + \sum_{j=1}^{N}\int_{G_j(\lambda,k)\Delta G_j(\lambda,0)}|H_j-H_{N+1}|dx\\ &\le \mathrm{Per}_\Phi(\mathcal{G}(\lambda,0)) + \sum_{j=1}^{N}\|H_j-H_{N+1}\|_{L^1(D)} := e_0, \quad \forall k\ge 1\end{aligned}$$

and

$$\bigcup_{j=1}^{N} G_j(\lambda,k) \subseteq D \qquad \forall \lambda \geq 1, \quad k\geq 1. \tag{4.21}$$

Fix $t, t' > 0$ with $0 < t - t' < 1$, $k_0 = [\lambda t']$, $m_0 = [\lambda t]$. Define

$$\theta := \min\Big\{\Big(\frac{\kappa}{2(N+2)\kappa_1 + 2N\kappa_2}\Big)^n, \Big(\frac{\kappa_1}{2(\kappa_1+\kappa_2)}\Big)^n\Big\}, \quad \gamma := \frac{\kappa n}{8N(\operatorname{diam} D + 2)}$$

and let λ be so large (depending on t, t', n, κ, H, Φ) that the density estimates in Proposition 4.14 holds for any $r \in (0, \frac{\gamma}{\lambda})$ and $m_0 \geq k_0 + 3 \geq 4$. We claim that

$$|\mathcal{G}(\lambda,[\lambda t])\Delta\mathcal{G}(\lambda,[\lambda t'])| \leq \mathrm{C}|t-t'|^{\frac{1}{n+1}} + \widetilde{\mathrm{C}}|t-t'|^{-\frac{n}{n+1}}\lambda^{-1}, \tag{4.22}$$

where

$$\mathrm{C} := \frac{3^n\omega_n^{1/n}\gamma}{\theta^{1-1/n}c_{\rm ball}(n)\kappa_1}\, e_0 + \frac{e_0}{\gamma} \quad \text{and} \quad \widetilde{\mathrm{C}} := \frac{3^n\omega_n^{1/n}\gamma}{\theta^{1-1/n}c_{\rm ball}(n)\kappa_1}\, e_0.$$

Since $k_0 \geq 1$, the entries of $\mathcal{G}(\lambda,k)$ satisfy the density estimates (4.15) in $(0,\gamma/\lambda)$, and hence from Proposition 4.14 applied with $\ell = \frac{\gamma}{\lambda}|t-t'|^{-\frac{1}{n+1}}$. we get

$$\begin{aligned}|\mathcal{G}(\lambda,[\lambda t])\Delta\mathcal{G}(\lambda,[\lambda t'])| \leq & \sum_{k=k_0+1}^{m_0} |\mathcal{G}(\lambda,k)\Delta\mathcal{G}(\lambda,k-1)| \\ \leq & \frac{3^n\omega_n^{1/n}\gamma}{\theta^{1-1/n}c_{\rm ball}(n)\kappa_1\lambda}\,|t-t'|^{-\frac{n}{n+1}} \sum_{k=k_0+1}^{m_0} \mathrm{Per}_{\Phi}(\mathcal{G}(\lambda,k-1)) \\ & + \frac{\lambda|t-t'|^{\frac{1}{n+1}}}{\gamma}\sum_{k=k_0+1}^{m_0}\sigma(\mathcal{G}(\lambda,k),\mathcal{G}(\lambda,k-1)).\end{aligned} \tag{4.23}$$

Using (4.20) the second sum is estimated as

$$\begin{aligned}\lambda\sum_{k=k_0+1}^{m_0}\sigma(\mathcal{G}(\lambda,k),\mathcal{G}(\lambda,k-1)) \leq & \mathrm{Per}_{\Phi}(\mathcal{G}(\lambda,0)) - \mathrm{Per}_{\Phi}(\mathcal{G}(\lambda,m_0)) \\ & + \int_{\mathcal{G}(\lambda,k)\Delta\mathcal{G}(\lambda,0)} |H_j - H_{N+1}|dx \leq e_0.\end{aligned}$$

Moreover, using $\mathrm{Per}_J(\mathcal{G}(\lambda,k-1)) \leq e_0$, from (4.23) we deduce

$$|\mathcal{G}(\lambda,[\lambda t])\Delta\mathcal{G}(\lambda,[\lambda t'])| \leq \frac{3^n\omega_n^{1/n}\gamma}{\theta^{1-1/n}c_{\rm ball}(n)\kappa_1\lambda}\,|t-t'|^{-\frac{n}{n+1}}\,(m_0-k_0)\,e_0 + \frac{e_0}{\gamma}\,|t-t'|^{\frac{1}{n+1}}.$$

Now (4.22) follows from $m_0 - k_0 \leq \lambda(t-t'+\frac{1}{\lambda})$.

Now we prove the assertions of the theorem. Using (4.21), the inequality $\mathrm{Per}\,(\mathcal{L}(\lambda,k)) \le e_0$, Proposition 4.9 and a diagonal argument we obtain the existence of a diverging sequence $\{\lambda_k\}$ and $\mathcal{M}(t) \in \mathbb{P}_b(N+1)$ such that

$$\lim_{k\to+\infty} |\mathcal{L}(\lambda_k,[\lambda_k t])\Delta\mathcal{M}(t)| = 0 \tag{4.24}$$

for every rational $t>0$ and also (4.19) holds. By (4.22) $\mathcal{M}(t)$ satisfies

$$|\mathcal{M}(t)\Delta\mathcal{M}(t')| \le \mathrm{C}|t-t'|^{\frac{1}{n+1}} \qquad \forall t',t \in \mathbb{Q}\cap(0,+\infty),\ |t-t'|<1.$$

Hence this map extends uniquely to a map $\{\mathcal{M}(t) : t>0\} \subseteq \mathbb{P}_b(N+1)$ satisfying (4.18) and (4.19).

It remains to show that $\mathcal{M} \in GMM(F,\mathcal{G})$. Since $\mathcal{L}(\lambda,0)=\mathcal{G}$, and we need just to prove (4.24) for any $t\ge 0$. Case $t=0$ is trivial: $\mathcal{M}(0)=\mathcal{G}$. Fix $t>0$. For every $\varepsilon\in(0,1)$ take $t_\varepsilon \in \mathbb{Q}\cap(0,+\infty)$ such that $|t-t_\varepsilon|<\varepsilon^{n+1}$. Since $\mathcal{M}$ satisfies (4.18), from (4.22) and (4.24) we deduce

$$\begin{aligned}\limsup_{k\to+\infty} |\mathcal{L}(\lambda_k,[\lambda_k t])\Delta\mathcal{M}(t)| \le& \limsup_{k\to+\infty} |\mathcal{L}(\lambda_k,[\lambda_k t])\Delta\mathcal{L}(\lambda_k,[\lambda_k t_\varepsilon])| \\ &+ \limsup_{k\to+\infty} |\mathcal{L}(\lambda_k,[\lambda_k t_\varepsilon])\Delta\mathcal{M}(t_\varepsilon)| + \limsup_{k\to+\infty} |\mathcal{M}(t_\varepsilon)\Delta\mathcal{M}(t)| \\ \le& 2\mathrm{C}|t-t_\varepsilon|^{\frac{1}{n+1}} \le 2\mathrm{C}\varepsilon.\end{aligned}$$

Hence, (4.24) is obtained letting $\varepsilon\to 0^+$.

Finally, let $\sum\limits_{j=1}^{N+1} |\overline{G_j}\setminus G_j| = 0$. Given $t\in(0,1)$, choosing λ sufficiently large, from (4.22) we get

$$\begin{aligned}|\mathcal{L}(\lambda,[\lambda t])\Delta\mathcal{L}(\lambda,0)| \le& \mathcal{L}(\lambda,[\lambda t])\Delta\mathcal{L}(\lambda,1)| + |\mathcal{L}(\lambda,1)\Delta\mathcal{G}| \\ \le& \mathrm{C}\left|t-\frac{1}{\lambda}\right|^{\frac{1}{n+1}} + \frac{\tilde{\mathrm{C}}}{\lambda|t-\frac{1}{\lambda}|^{\frac{n}{n+1}}} + |\mathcal{L}(\lambda,1)\Delta\mathcal{G}|.\end{aligned}$$

Now letting $\lambda\to+\infty$ and using Proposition 4.12 a) we establish

$$|\mathcal{M}(t)\Delta\mathcal{M}(0)| \le \mathrm{C}\, t^{\frac{1}{n+1}}.$$

□

Corollary 4.17. *Let $H_j \equiv 0$, $j=1,\dots,N+1$. Then any $\mathcal{M}\in GMM(F_H^\Phi,\mathcal{G})$ is locally uniformly $1/(n+1)$-Hölder continuous and*

$$\bigcup_{j=1}^{N} M_j(t) \subseteq \mathrm{co}(\mathcal{G}) \qquad \forall t\ge 0.$$

In order to improve the Hölder exponent $\frac{1}{n+1}$ to the value $\frac{1}{2}$ in (4.18) we expect to be useful, for minimizers $\mathcal{A}(\lambda)$ of $F_H^\Phi(\cdot,\mathcal{A};\lambda)$, an estimate of the form

$$\sum_{i=1}^{N+1} \sup_{A_i(\lambda)\Delta A_i} d(\cdot,\partial A_i) \le O(\lambda^{-1/2}).$$

We miss the proof of such an estimate; however, a partial result in this direction is given in Lemma 4.21.

4.3 Uniqueness and consistency of GMM for convex disjoint partitions

In this section we suppose that the entries of Φ are Euclidean and $H\equiv 0$. For shortness of notation we write $F_H^\Phi := F$.

Definition 4.18 (Convex and disjoint partitions). *A partition $\mathcal{A}\in\mathbb{P}_b(N+1)$ is called convex if the bounded components of $\mathcal{A}$ are convex and is called disjoint provided*

$$\min_{1\le i<j\le N} \operatorname{dist}(A_i,A_j) > 0.$$

Notice that if $\mathcal{A}\in\mathbb{P}_b(N+1)$ is disjoint, then $\operatorname{Per}(\mathcal{A}) = 2\sum_{j=1}^{N} P(A_j)$. Moreover, if $\mathcal{A}$ and $\mathcal{G}$ are disjoint and satisfy

$$\bigcup_{j=1}^{N}(A_j\Delta G_j) = \Big(\bigcup_{j=1}^{N} A_j\Big)\Delta\Big(\bigcup_{j=1}^{N} G_j\Big), \tag{4.25}$$

then $\sigma(\mathcal{A},\mathcal{G}) = \sum\limits_{j=1}^{N+1}\int_{A_j\Delta G_j} d(x,\partial G_j)dx$ and thus

$$F(\mathcal{A},\mathcal{G};\lambda) = 2\sum_{j=1}^{N}\Big(P(A_j)+\lambda\int_{A_j\Delta G_j} d(x,\partial G_j)dx\Big). \tag{4.26}$$

The aim of this section is to prove the following consistency result.

Theorem 4.19 (Evolution of convex disjoint partitions). *Assume that $\mathcal{C}\in\mathbb{P}_b(N+1)$ is disjoint and convex. Then*

$$GMM(F,\mathcal{C}) = \{\mathcal{M}\} = \{(M_1,\dots,M_{N+1})\}$$

is a singleton. Moreover, for any $i=1,\dots,N$, $M_i(\cdot)$ agrees with the classical mean curvature flow starting from C_i up to its extinction time.

In particular, for any $i,j\in\{1,\dots,N\}$, $i\ne j$, the function

$$t\in[0,\min\{t_i^\dagger,t_j^\dagger\})\mapsto \operatorname{dist}(M_i(t),M_j(t))$$

is nondecreasing, where $t_h^\dagger$ is the extinction time of C_h [66].

We postpone the proof of this theorem after several auxiliary results. The proof of the following lemma is an adaptation of the proof of Theorem 4.6.

Lemma 4.20. *Given $\mathcal{G} \in \mathbb{P}_b(N+1)$ let $\mathcal{G}(\lambda) \in \mathbb{P}_b(N+1)$ be a minimizer of $F(\cdot, \mathcal{G}; \lambda)$. Fix $i \in \{1, \dots, N+1\}$. If $x \in G_i(\lambda)^c \cap G_i$ and $d(x, \partial G_i) \geq \rho > 0$, then*

$$\frac{1}{2^n} \leq \frac{|B_\rho(x) \cap G_i(\lambda)^c|}{|B_\rho(x)|}. \tag{4.27}$$

Proof. Since the idea of the proof is the same for any i, we suppose $i = 1$. As usual, write $B_r := B_r(x)$ and set

$$I := \{j \in \{2, \dots, N+1\} : \ \mathcal{H}^{n-1}(B_\rho \cap \partial^* G_1(\lambda) \cap \partial^* G_j(\lambda)) > 0\}.$$

Clearly, if $I = \emptyset$, then by Remark 4.4 $B_\rho \subseteq G_1(\lambda)^c$ and (4.27) is satisfied, hence we can suppose $I \neq \emptyset$. Fix any $r \in (0, \rho)$ such that

$$\sum_{j=1}^{N+1} \mathcal{H}^{n-1}(\partial B_r \cap \partial^* G_j(\lambda)) = 0. \tag{4.28}$$

For each $j \in I$ define the competitor $\mathcal{B} \in \mathbb{P}_b(N+1)$ as

$$\mathcal{B} := (G_1(\lambda) \cup (G_j(\lambda) \cap B_r), G_2(\lambda) \dots, G_{j-1}(\lambda), G_j(\lambda) \setminus B_r, G_{j+1}(\lambda), \dots, G_{N+1}(\lambda)). \tag{4.29}$$

Fix $s \in (r, \rho)$; since

$$\begin{aligned} P(G_1(\lambda) \cup (G_j(\lambda) \cap B_r), B_s) =& P(G_1(\lambda), B_s) + \mathcal{H}^{n-1}(G_j(\lambda) \cap \partial B_r) + P(G_j(\lambda), B_r) \\ & - 2\mathcal{H}^{n-1}(B_r \cap \partial^* G_1(\lambda) \cap \partial^* G_j(\lambda)), \\ P(G_j(\lambda) \setminus B_r, B_s) =& P(G_j(\lambda), B_s \setminus \overline{B_r}) + \mathcal{H}^{n-1}(G_j(\lambda) \cap \partial B_r), \end{aligned}$$

from (4.28)

$$\begin{aligned} \lim_{s \to r^+} \Big(& P(G_1(\lambda) \cup (G_j(\lambda) \cap B_r), B_s) + P(G_j(\lambda) \setminus B_r, B_s) - P(G_1(\lambda), B_s) - P(G_j(\lambda), B_s) \Big) \\ &= 2\mathcal{H}^{n-1}(G_j(\lambda) \cap \partial B_r) - 2\mathcal{H}^{n-1}(B_r \cap \partial^* G_1(\lambda) \cap \partial^* G_j(\lambda)). \end{aligned}$$

Now the minimality of $\mathcal{G}(\lambda)$ and (4.9) imply

$$\begin{aligned} \mathcal{H}^{n-1}(G_j(\lambda) \cap \partial B_r) - & \mathcal{H}^{n-1}(B_r \cap \partial^* G_1(\lambda) \cap \partial^* G_j(\lambda)) \\ & \geq \frac{\lambda}{2} \int_{G_j(\lambda) \cap B_r} \big(\tilde{d}(y, \partial G_j) - \tilde{d}(y, \partial G_1)\big) dy. \end{aligned} \tag{4.30}$$

Since $B_\rho \subseteq G_1$ (and hence $B_\rho \cap G_j = \emptyset$) we have

$$\tilde{d}(y, \partial G_j) - \tilde{d}(y, \partial G_1) = d(y, \partial G_j) + d(y, \partial G_1) \geq 0 \qquad \forall y \in G_j(\lambda) \cap B_r,$$

and therefore

$$\mathcal{H}^{n-1}(B_r \cap \partial^* G_1(\lambda) \cap \partial^* G_j(\lambda)) \leq \mathcal{H}^{n-1}(G_j(\lambda) \cap \partial B_r). \tag{4.31}$$

Then summation of (4.31) over $j \in I$ and use of Remark 4.4 yield

$$P(G_1(\lambda)^c, B_r) \le \sum_{j\in I} \mathcal{H}^{n-1}(G_j(\lambda) \cap \partial B_r) \le \sum_{j=2}^{N+1} \mathcal{H}^{n-1}(G_j(\lambda) \cap \partial B_r) = \mathcal{H}^{n-1}(G_1(\lambda)^c \cap \partial B_r).$$

Now adding $\mathcal{H}^{n-1}(G_1(\lambda)^c \cap \partial B_r)$ to both sides we get

$$P(G_1(\lambda)^c \cap B_r) \le 2\mathcal{H}^{n-1}(G_1(\lambda)^c \cap \partial B_r).$$

From the isoperimetric inequality, for a.e. $r \in (0, \rho)$ we obtain

$$n\omega_n^{1/n} |G_1(\lambda)^c \cap B_r|^{\frac{n-1}{n}} \le 2\mathcal{H}^{n-1}(G_1(\lambda)^c \cap \partial B_r). \tag{4.32}$$

Since $x \in G_1(\lambda)^c$, one has $|G_1(\lambda)^c \cap B_r| > 0$ for any $r > 0$, therefore integrating (4.32) in $(0, \rho)$, we get (4.27). □

Lemma 4.21. *Given $\mathcal{G} \in \mathbb{P}_b(N+1)$ let $\mathcal{G}(\lambda) \in \mathbb{P}_b(N+1)$ be a minimizer of $F(\cdot, \mathcal{G}; \lambda)$. Then for any $i \in \{1, \dots, N+1\}$,*

$$\sup_{x \in G_i(\lambda)^c \cap G_i} d(x, \partial G_i) \le \frac{\sqrt{2^{n+2}n}}{\sqrt{\lambda}}.$$

Proof. Without loss of generality we suppose $i = 1$. By contradiction, let $x \in G_1(\lambda)^c \cap G_1$ be such that $d(x, \partial G_1) \ge \rho := \frac{\sqrt{2^{n+2}n} + \varepsilon}{\sqrt{\lambda}}$ for some $\varepsilon > 0$. Possibly decreasing ε we may suppose that $x \in \partial G_1(\lambda)$, and ρ satisfies (4.28) with $r = \rho$, so that the set

$$J := \{j \in \{2, \dots, N+1\} : \ |B_{\rho/2} \cap G_j(\lambda)| > 0\}$$

is nonempty, $B_{\rho/2} := B_{\rho/2}(x)$. Moreover for every $y \in B_{\rho/2}$, the ball centered at y of radius $\rho/2$ is contained in G_1 and hence

$$d(y, \partial G_j) \ge d(y, \partial G_1) \ge \rho/2 \qquad \forall j \in J.$$

Therefore, for each $j \in J$ defining the competitor as in (4.29) with $r = \rho/2$, from the minimality of $\mathcal{G}(\lambda)$, (4.9) and (4.30) we get

$$\begin{aligned}\mathcal{H}^{n-1}&(G_j(\lambda) \cap \partial B_{\rho/2}) - \mathcal{H}^{n-1}(B_{\rho/2} \cap \partial^* G_1(\lambda) \cap \partial^* G_j(\lambda)) \\ &\ge \frac{\lambda}{2} \int_{G_j(\lambda)\cap B_{\rho/2}} \big(\tilde{d}(y, \partial G_j) - \tilde{d}(y, \partial G_1)\big) dy \ge \frac{\lambda\rho}{2} |G_j(\lambda) \cap B_{\rho/2}|,\end{aligned}$$

since $\tilde{d}(y, \partial G_j) = d(y, \partial G_j)$ and $\tilde{d}(y, \partial G_1) = -d(y, \partial G_1)$ for any $y \in B_{\rho/2}$. Summing these inequalities over $j \in J$ and using $\bigcup_{j=1}^{N+1} (G_j(\lambda) \cap B_{\rho/2}) = \bigcup_{j\in J} (G_j(\lambda) \cap B_{\rho/2}) = G_1(\lambda)^c \cap B_{\rho/2}$ (up to a negligible set), we get

$$\mathcal{H}^{n-1}(G_1(\lambda)^c \cap \partial B_{\rho/2}) \ge \sum_{j\in J} \mathcal{H}^{n-1}(B_{\rho/2} \cap \partial^* G_1(\lambda) \cap \partial^* G_j(\lambda)) + \frac{\lambda\rho}{2} |G_1(\lambda)^c \cap B_{\rho/2}|.$$

Now Lemma 4.20 yields

$$\left(\frac{1}{2}\right)^{n+1}\lambda\rho\omega_n\left(\frac{\rho}{2}\right)^n \le \mathcal{H}^{n-1}(G_1(\lambda)^c \cap \partial B_{\rho/2}) \le n\omega_n\left(\frac{\rho}{2}\right)^{n-1}.$$

But this implies $\rho = \frac{\sqrt{2^{n+2}n}+\varepsilon}{\sqrt{\lambda}} \le \frac{\sqrt{2^{n+2}n}}{\sqrt{\lambda}}$, a contradiction, since $\varepsilon > 0$. □

Given $A \subseteq \mathbb{R}^n$ and $\delta > 0$ set

$$A_\delta^+ := \{x \in \mathbb{R}^n : \ \mathrm{dist}(x, A) \le \delta\}.$$

The following theorem, valid without any convexity assumption on the components, shows that if the entries of the initial partition $\mathcal{G}$ are far from each other, then so are the entries of minimizers of $F(\cdot, \mathcal{G}; \lambda)$ provided λ is large.

Theorem 4.22 (Minimizers of F for a disjoint initial partition). *Suppose that $\mathcal{G} \in \mathbb{P}_b(N+1)$ is disjoint and set*

$$\min_{1\le i<j\le N} \mathrm{dist}(G_i, G_j) =: \varepsilon_0 > 0. \tag{4.33}$$

Then for $\lambda \ge 2^{n+6}n\varepsilon_0^{-2}$ any minimizer $\mathcal{G}(\lambda)$ of $F(\cdot, \mathcal{G}; \lambda)$ satisfies

$$G_j(\lambda) \subseteq (G_j)^+_{\varepsilon_0/4}, \qquad j = 1, \dots, N.$$

Proof. We claim that the choice of λ implies

$$G_{N+1}(\lambda)^c \subseteq (G^c_{N+1})^+_{\varepsilon_0/4}. \tag{4.34}$$

Indeed, obviously $G_{N+1}(\lambda)^c \cap G^c_{N+1} \subseteq (G^c_{N+1})^+_{\varepsilon_0/4}$. Now if $x \in G_{N+1}(\lambda)^c \cap G_{N+1}$, then $d(x, G^c_{N+1}) = d(x, \partial G_{N+1})$ and therefore by Lemma 4.21

$$d(x, G^c_{N+1}) \le \sup_{y \in G_{N+1}(\lambda)^c \cap G_{N+1}} d(y, \partial G_{N+1}) \le \frac{\sqrt{2^{n+2}n}}{\sqrt{\lambda}} \le \frac{\varepsilon_0}{4}.$$

Hence $x \in (G^c_{N+1})^+_{\varepsilon_0/4}$.

We prove the assertion of the theorem arguing by contradiction. Suppose for example $j = 1$ and $G_1(\lambda)$ is not contained in $(G_1)^+_{\varepsilon_0/4}$. In view of (4.34) and (4.33)

$$G_1(\lambda) \subseteq \bigcup_{j=1}^N G_j(\lambda) \subseteq \Big(\bigcup_{j=1}^N G_j\Big)^+_{\varepsilon_0/4} = \bigcup_{j=1}^N (G_j)^+_{\varepsilon_0/4}.$$

Since our assumption implies $G_1(\lambda) \cap (G_j)^+_{\varepsilon_0/4} \ne \emptyset$ for some $j \in \{2, \dots, N\}$, and by virtue of Remark 4.15 the set $G_1(\lambda)$ can be supposed to be open, there exists a ball B_r of radius $r > 0$ whose closure is contained in $G_1(\lambda) \cap (G_j)^+_{\varepsilon_0/4}$. For shortness, let $j = 2$. Thus setting $\mathcal{B} := (G_1(\lambda) \setminus B_r, G_2(\lambda) \cup B_r, G_3(\lambda), \dots, G_{N+1}(\lambda))$, and using $P(G_1(\lambda)) - P(G_1(\lambda) \setminus B_r) = P(B_r)$, we obtain

$$\begin{aligned} 2F(\mathcal{G}(\lambda), \mathcal{G}; \lambda) - 2F(\mathcal{B}, \mathcal{G}; \lambda) =& P(B_r) + P(G_2(\lambda)) - P(G_2(\lambda) \cup B_r) \\ &+ \lambda \int_{B_r} \big(\tilde{d}(x, \partial G_1) - \tilde{d}(x, \partial G_2)\big)\, dx. \end{aligned}$$

Since $B_r \cap G_2(\lambda) = \emptyset$, from (1.2) we get

$$P(B_r) + P(G_2(\lambda)) - P(B_r \cup G_2(\lambda)) \geq 0.$$

In addition, by the definition of ε_0, $d(B_r, G_1) \geq \frac{3\varepsilon_0}{4}$, (thus $\tilde{d}(\cdot, \partial G_1) = d(\cdot, \partial G_1)$ in B_r); moreover, since $B_r \subseteq (G_2)^+_{\varepsilon_0/4}$, one has

$$\tilde{d}(x, \partial G_1) - \tilde{d}(x, \partial G_2) \geq \frac{\varepsilon_0}{4} \qquad \forall x \in B_r$$

and therefore

$$F(\mathcal{G}(\lambda), \mathcal{G}; \lambda) - F(\mathcal{B}, \mathcal{G}; \lambda) \geq \frac{\lambda\varepsilon_0}{8}|B_r| > 0.$$

This implies that $\mathcal{G}(\lambda)$ is not a minimizer of $F(\cdot, \mathcal{G}; \lambda)$. □

Corollary 4.23. *Suppose that $\mathcal{G} \in \mathbb{P}_b(N+1)$ is disjoint. Then for sufficiently large λ, $\mathcal{G}(\lambda)$ is a minimizer of $F(\cdot, \mathcal{G}; \lambda)$ if and only if each bounded component $G_j(\lambda)$, $j = 1, \dots, N$, of $\mathcal{G}(\lambda)$ is a minimizer of the Almgren-Taylor-Wang functional (4.13) with G replaced by G_j.*

Proof. Let

$$\min_{1 \leq i < j \leq N} \mathrm{dist}(G_i, G_j) =: \varepsilon_0 > 0.$$

Suppose that A_j, $j = 1, \dots, N$, minimizes (4.13) with G replaced by G_j. By [78, Lemma 2.1] (see also [19, Proposition 5.5]) there exists $c(n) > 0$ such that

$$\sup_{x \in A_j \Delta G_j} d(x, \partial G_j) \leq \sqrt{\frac{c(n)}{\lambda}}.$$

Therefore, taking

$$\lambda \geq \tilde{c}(n)\varepsilon_0^{-2}, \quad \tilde{c}(n) := \max\{2^{n+6}n, 16c(n)\}, \tag{4.35}$$

we deduce $A_j \subseteq (G_j)^+_{\varepsilon_0/4}$, $j = 1, \dots, N$. Set $\mathcal{A} = (A_1, \dots, A_N, \mathbb{R}^n \setminus \bigcup_{j=1}^N A_j)$. Let us show that for λ as in (4.35), $\mathcal{A}$ minimizes $F(\cdot, \mathcal{G}; \lambda)$. Indeed, take any minimizer $\mathcal{G}(\lambda)$ of $F(\cdot, \mathcal{G}; \lambda)$. By Theorem 4.22 we have $G_j(\lambda) \subseteq (G_j)^+_{\varepsilon_0/4}$, therefore both $(\mathcal{A}, \mathcal{G})$ and $(\mathcal{G}(\lambda), \mathcal{G})$ satisfy (4.25). Hence, (4.26) and the minimality of A_j yield

$$\begin{aligned} F(\mathcal{G}(\lambda), \mathcal{G}; \lambda) &= \sum_{j=1}^N \Big(P(G_j(\lambda)) + \lambda \int_{G_j(\lambda)\Delta G_j} d(x, \partial G_j)dx \Big) \\ &\geq \sum_{j=1}^N \Big(P(A_j) + \lambda \int_{A_j \Delta G_j} d(x, \partial G_j)dx \Big) = F(\mathcal{A}, \mathcal{G}; \lambda). \end{aligned}$$

This implies that $\mathcal{A}$ is also a minimizer $F(\cdot, \mathcal{G}; \lambda)$.

Conversely, suppose that λ satisfies (4.35) and $\mathcal{G}(\lambda)$ minimizes $F(\cdot, \mathcal{G}; \lambda)$ and let A_j, $j = 1, \dots, N$, be a minimizer (4.13) with G replaced by G_j. Recall that $A_j \subseteq (G_j)^+_{\varepsilon_0/4}$, $j = 1, \dots, n$.

Set $\mathcal{A}=(A_1,\dots,A_N,\mathbb{R}^n\setminus\bigcup_{j=1}^N A_j)$. Then from the minimality of A_j and $\mathcal{G}(\lambda)$, as well as (4.26), we deduce

$$F(\mathcal{G}(\lambda),\mathcal{G};\lambda)\le F(\mathcal{A},\mathcal{G};\lambda)=\sum_{j=1}^N\Big(P(A_j)+\lambda\int_{A_j\Delta G_j}d(x,\partial G_j)dx\Big)$$

$$\le\sum_{j=1}^N\Big(P(G_j(\lambda))+\lambda\int_{G_j(\lambda)\Delta G_j}d(x,\partial G_j)dx\Big)=F(\mathcal{G}(\lambda),\mathcal{G};\lambda).$$

Thus all inequalities are in fact equalities, which is possible if and only if

$$P(G_j(\lambda))+\lambda\int_{G_j(\lambda)\Delta G_j}d(x,\partial G_j)=P(A_j)+\lambda\int_{A_j\Delta G_j}d(x,\partial G_j)dx,\quad j=1,\dots,N.$$

Hence, $G_j(\lambda)$ is a minimizer of (4.13) with $G=G_j$. □

Proof of Theorem 4.19. Suppose that

$$\min_{1\le i<j\le N}\operatorname{dist}(C_i,C_j)\ge\varepsilon_0>0. \tag{4.36}$$

By [17, Corollary 5] the Amgren-Taylor-Wang solution $M_i(\cdot)$ starting from C_i (i.e. GMM starting from C_i and associated with (4.13)), $i=1,\dots,N$, is unique and agrees with the classical mean curvature flow starting from C_i up to its extinction time. Moreover, since $M_i(\cdot)\subseteq C_i$, for any $t\ge0$ we have $\mathcal{M}(t):=(M_1(t),\dots,M_N(t),\mathbb{R}^n\setminus\bigcup_{i=1}^N M_i(t))\in\mathbb{P}_b(N+1)$.

We claim that $GMM(F,\mathcal{C})=\{\mathcal{M}\}$.

Indeed, let $\mathcal{C}(\lambda)\in\mathbb{P}_b(N+1)$ be a minimizer of $F(\cdot,\mathcal{C};\lambda)$. By Corollary 4.23 if λ satisfies (4.35), then $C_i(\lambda)$ minimizes the Almgren-Taylor-Wang functional (4.13) with $G=C_i$. By [17, Remark 8], $C_i(\lambda)\subseteq C_i$, $C_i(\lambda)$ is convex. Hence, $\mathcal{C}(\lambda)$ also satisfies (4.36).

Define $\mathcal{C}(\lambda,k)$ as $\mathcal{C}(\lambda,0)=\mathcal{C}$ and

$$F(\mathcal{C}(\lambda,k),\mathcal{C}(\lambda,k-1);\lambda)=\min_{\mathcal{A}\in\mathbb{P}_b(N+1)}F(\mathcal{A},\mathcal{C}(\lambda,k-1);\lambda).$$

From the previous observation, for λ satisfies (4.35) and $k\ge1$ each $C_i(\lambda,k)$, $i=1,\dots,N$, is a minimizer of (4.13) with $G=C_i(\lambda,k-1)$. Therefore, by [17, Corollary 5]

$$\lim_{\lambda\to+\infty}|C_i(\lambda,[\lambda t])\Delta M_i(t)|=0,\qquad\forall t\ge0,\ i=1,\dots,N. \tag{4.37}$$

Since $C_i(\lambda,[\lambda t]),M_i(t)\subseteq C_i$, $i=1,\dots,N$, from (4.37) we deduce

$$\lim_{\lambda\to+\infty}|\mathcal{C}(\lambda,[\lambda t])\Delta\mathcal{M}(t)|=\lim_{\lambda\to+\infty}2\sum_{i=1}^N|C_i(\lambda,[\lambda t])\Delta M_i(t)|=0$$

for any $t\ge0$. Thus, $GMM(F,\mathcal{C})=\{\mathcal{M}\}$.

□

Theorem 4.24 (Stability of convex disjoint partitions). *Under the hypotheses of Theorem 4.19, if the sequence* $\{\mathcal{G}^{(h)}\} \subset \mathbb{P}_b(N+1)$ *converges to* $\mathcal{C}$ *in the Hausdorff distance* $\mathbb{HD}$ *as* $h \to +\infty$*, then for any* $\mathcal{M}^{(h)} \in GMM(F, \mathcal{G}^{(h)})$*,*

$$\lim_{h\to+\infty} \mathbb{HD}(\mathcal{M}^{(h)}(t), \mathcal{M}(t)) := \lim_{h\to+\infty} \sum_{i=1}^{N} \mathbb{HD}(M_i^{(h)}(t), M_i(t)) = 0 \qquad \forall t \in [0, \min_{i\le N} t_i^\dagger),$$

where $t_i^\dagger$ *is the extinction time of* C_i*.*

Proof. Let us show first the following comparison principle:

Claim 1. If $\mathcal{C} \in \mathbb{P}_b(N+1)$ is convex and satisfies

$$\min_{1\le i<j\le N} \operatorname{dist}(C_i, C_j) \ge \varepsilon_0 > 0. \tag{4.38}$$

then for every $\mathcal{G} \in \mathbb{P}_b(N+1)$ with $G_i \subseteq C_i$, $i = 1, \dots, N$, for every minimizer $\mathcal{G}(\lambda)$ of $F(\cdot, \mathcal{G}; \lambda)$, the inclusion $G_i(\lambda) \subseteq C_i$ holds provided $\lambda \ge \tilde{c}(n)\varepsilon_0^{-2}$. In particular, $\mathcal{G}(\lambda)$ also satisfies (4.38) unless $G_i(\lambda) = \emptyset$.

Indeed, let $C_i(\lambda)^*$, $i = 1, \dots, N$ be the maximal minimizer [19, Definition 6.4] of the Almgren-Taylor-Wang functional (4.13) with $G = C_i$. By [17, Remark 8] $C_i(\lambda)^* \subseteq C_i$, and from Corollary 4.23

$$\mathcal{C}(\lambda) = \Big(C_1(\lambda), \dots, C_N(\lambda), \mathbb{R}^n \setminus \bigcup_{i=1}^{N} C_i(\lambda)\Big)$$

is a minimizer of $F(\cdot, \mathcal{C}; \lambda)$. Since $\mathcal{G}$ also satisfies (4.38), by Corollary 4.23 each $G_i(\lambda)$, $i = 1, \dots, N$ is a minimizer of (4.13). Then by [19, Theorem 6.1] one has $G_i(\lambda) \subseteq C_i(\lambda)^* \subseteq C_i$ for any $i \le N$.

Now we show the following stability property of convex sets.

Claim 2. Let $C \subset \mathbb{R}^n$ be a nonempty bounded convex set and a sequence of sets of finite perimeter $G^{(h)}$ converge to C in Hausdorff distance as $h \to +\infty$. Then

$$G^{(h)}(t) \overset{\mathbb{HD}}{\to} C(t), \quad t \in [0, t_C^\dagger), \tag{4.39}$$

where $G^{(h)}(t)$ and $C(t)$ are Almgren-Taylor-Wang solutions starting from $G^{(h)}$ and C respectively (recall that $C(\cdot)$ is unique by [17, Corollary 5]), and $t_C^\dagger$ is the extinction time of C.

Indeed, consider arbitrary sequences $\{A^{(l)}\}$, $\{B^{(l)}\}$ of convex sets such that $A^{(l)} \subset\subset C \subset\subset B^{(l)}$, $l \ge 1$, and $A^{(l)}, B^{(l)} \overset{\mathbb{HD}}{\to} C$ as $l \to +\infty$. Then for any $l \ge 1$, there exists $h_l > 0$ such that $A^{(l)} \subseteq C^{(h)} \subseteq B^{(l)}$ for any $h > h_l$. Let $A^{(l)}(t)$ (resp. $B^{(l)}(t)$) be the minimizing movements starting from $A^{(l)}$ (resp. $B^{(l)}$) for the Almgren-Taylor-Wang functional (4.13) and $G^{(h)}(t)^*$ and $G^{(h)}(t)_*$ be the maximal and minimal GMMs [19, Definition 7.2] for (4.13) starting from $G^{(h)}$ and so that $G^{(h)}{}_*(t) \subseteq G^{(h)}(t) \subseteq G^{(h)*}(t)$ for all $t \ge 0$. By the comparison theorem [19, Theorem 7.3], $A^{(l)}(t) \subseteq G^{(h)}{}_*(t)$ and $G^{(h)*}(t) \subseteq B^{(l)}(t)$ for any $t \ge 0$. Moreover, from [17, Theorem 12] we have $A^{(l)}(t), B^{(l)}(t) \overset{\mathbb{HD}}{\to} C(t)$ as $l \to +\infty$ for any $t \in [0, t_C)$, and since $h_l \to +\infty$, (4.39) follows.

Now we prove the assertion of the theorem. Let $\mathcal{A} \in \mathbb{P}_b(N+1)$ be a convex disjoint partition with $C_i \subset\subset A_i$, $i = 1, \dots, N$. Then for sufficiently large h, $G_i^{(h)} \subset A_i$. Let $\mathcal{G}^{(h)}(\lambda_{h,k}, [\lambda_{h,k}t])$ be the sequence chosen in the definition of $\mathcal{M}^{(h)}(t)$, i.e. $\mathcal{G}^{(h)}(\lambda_{h,k}, [\lambda_{h,k}t])$ minimizes $F(\cdot, \mathcal{G}^{(h)}(\lambda_{h,k}, [\lambda_{h,k}t] - 1); \lambda_{h,k})$ and $\mathcal{G}^{(h)}(\lambda_{h,k}, [\lambda_{h,k}t]) \to \mathcal{M}^{(h)}(t)$ in $L^1(\mathbb{R}^n)$ as $k \to +\infty$. By Claim 1 and Corollary 4.23, each $G_i^{(h)}(\lambda_{h,k}, [\lambda_{h,k}t])$, $i = 1, \dots, N$ minimizes (4.13) with $G = G_i^{(h)}(\lambda_{h,k}, [\lambda_{h,k}t] - 1)$, therefore, $M_i^{(h)}(\cdot)$ is an Almgren-Taylor-Wang solution starting from $G_i^{(h)}$. Now as $G_i^{(h)} \overset{\mathbb{HD}}{\to} C_i$, Claim 2 implies $M_i^{(h)}(t) \overset{\mathbb{HD}}{\to} M_i(t)$, $i = 1, \dots, N$ as $h \to +\infty$ for any $t \in [0, t_i^{\dagger})$. $\square$

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